# ANALYTIC METHODS

# FOR TACTICAL AIR WARFARE—2005

## AIR CAMPAIGN AND MATHEMATICAL ANALYSIS

REPORT PA406T1

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# Chapter 1
# Introduction

The Office of the Secretary Defense, Program Analysis and Evaluation, Tactical Air Division (OSD/PA&E TACAIR) has an ongoing effort focused on modeling and analysis of air combat and air campaigns in support of OSD decision making and planning. LMI has been assisting TACAIR with that effort. Our previous tasks have included air-to-air campaign modeling and laser propagation modeling:

◆ *Air-to-air campaign modeling.* LMI developed extensions to, and applied, the Stochastic Lanchester Air-to-Air Combat Model (SLAACM). We also completed several related mathematical and campaign analyses. The original version of SLAACM, which includes a single Blue defender against an array of Red escorts and bombers, was expanded to include multiple Blue defender types for two different cases. In the first case, the model was extended to include sequential defense by different types of Blue aircraft. In the second case, the model was extended to include an optimized defense using a mix of Blue defender types simultaneously. These resulted in two different versions of the model. The sequential defense model has been documented as an update version, but the simultaneous defense version has not yet been formally documented.[1]

◆ *Laser propagation modeling.* LMI extended the Airborne Laser Infrared Transmission model to include 1.06 μ wavelength solid-state lasers.[2] In the process, we developed a new approach that allows straightforward inclusion of additional laser wavelengths as needed.

The primary focus of the current task was air campaign modeling, both with SLAACM and with separate mathematical analyses. We used all versions of SLAACM in our classified analysis facility to address specific scenarios of interest to TACAIR. The classified results have been reported to TACAIR separately.

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[1] Current work on the simultaneous model includes linking to an integer optimizer to corroborate or replace the current heuristic optimizer.

[2] 1.06 μ is the wavelength of a neodymium-doped yttrium aluminum garnet (Nd:YAG) laser. We previously modeled the 3.152 μ chemical oxygen-iodine laser (COIL). The symbol, μ, stands for micron or 1 millionth ($10^{-6}$) meter. Wavelength ($\lambda$) is also expressed in units of Angstroms (Å) ($10^{-10}$ meter), and nanometers ($10^{-9}$ meter). The wavelength can alternatively be expressed as a wave number that is equal to $1/\lambda$ with units of inverse centimeters ($cm^{-1}$).

In addition to pure air-to-air campaign analyses using SLAACM, we analyzed general problems of ship interdiction and defense, suppression of enemy air defense, and runway interdiction. This report conveys the results of that aspect of our work:

- Chapter 2 contains a description of the sequential defense version of SLAACM, plus discussions of several individual campaign-related analyses.

- Chapter 3 is a discussion of an investigation into using SLAACM as a component in integrated analysis.

- Chapter 4 documents a mathematical analysis of runway interdiction by cluster bombs.

# Chapter 2
# Campaign Analyses

In this chapter, we describe mathematical models of engagements between tactical air forces and certain land and sea-based forces. We describe the sequential defense version of SLAACM. We develop an engagement model to account for engagements of tactical aircraft when one combatant's fighters carry only two missiles. We discuss generalized activity network models of the sequences of events that may occur when a Red force, based in a wide geographical region, attacks a much smaller defended region. We derive relationships and spreadsheet methods for calculating the probability, for a given probability of success for a single draw, and assuming replacement after each draw, of obtaining at least one of each of *M* possible outcomes in a sequence of *N* draws. Last, we analyze the relationship between kill-rates and exchange ratios. The material is presented in this order:

- ◆ SLAACM with sequential defense

- ◆ New models of escorted bombers engaged by surface-to-air missile (SAM) sites

- ◆ Models of engagements between tactical suppression of enemy air defense (SEAD)  aircraft and missile-firing ships

- ◆ Model of tactical aircraft engagements when one aircraft type carries only two missiles

- ◆ Modeling of combined-force engagements with generalized activity networks

- ◆ Probability, given M different items and replacement after each draw, of drawing at least one of each of the M possible outcomes in N draws

- ◆ Kill-rates and exchange ratios.

## SEQUENTIAL DEFENSE

We developed an extension of SLAACM to treat a case in which the attackers are intercepted by two waves of defenders. In this treatment, the attackers first encounter the Black defending force. Attackers leaking from engagements with the Black force encounter the Green defending force.

We assumed that the Red leakers were able to regroup, after their encounters with the Black force, into an optimal configuration for the encounters with the Green force. This assumption is, of course, optimistic for Red. In later work, we intend to add the case in which Red packages have a return-to-base (RTB) or breakaway criterion, such as "RTB if more than four escorts are destroyed." In this case, the Green force would encounter only those attack packages that did not return to base.

We implemented the extension by adding a driver subroutine, and worksheets for inputs and outputs for the sequential-defense case, to SLAACM.

To use the extended version of SLAACM, the user enters the initial Red order of battle in the SLAACM Main worksheet as for a regular SLAACM run. Then, on Sheet 7, the user enters the Black aircraft type, the number of Black aircraft, the number of Green aircraft, and the number of "days" that the campaign is to run. The Black aircraft can be any Blue_NFX aircraft. The Green aircraft is always type LAD. (These restrictions on Black and Green aircraft types can be easily relaxed.)

When those data are entered, the user executes the subroutine "run_sequential." That subroutine operates the campaign in this way: for each day, Red launches an optimal dispatch of attack packages against that day's Black force. As in previous SLAACM versions, Black responds with foreknowledge of Red's packages' compositions, or not, as the user specifies on SLAACM's Main worksheet.

After the Red/Black engagements, Red regroups into optimal dispatch options against the Green force, and the Red/Green engagements take place.

The subroutine "run_sequential" collects "daily" strengths for all Red, Blue, and Green aircraft on Sheet 7. Each day has two sets of engagements, and Sheet 7 shows the status of Red, Black, and Green forces after Black and Green engagements separately. Sheet 8 gives charts of Red, Black, and Green orders of battle as they evolve during the campaign. Figure 2-1 and Figure 2-2 show orders of battle for the case in which the Black force is 48 Blue_NF2 fighters, the Green force is 200 LADs, and the Red force is the standard SLAACM Red force.

*Figure 2-1. Evolution of Red Order of Battle*
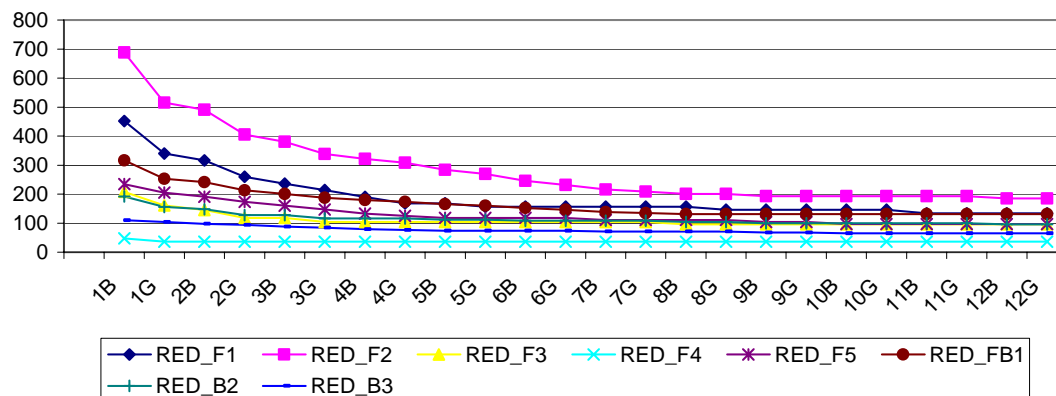


*Figure 2-2. Evolution of Defenders' Order of Battle*
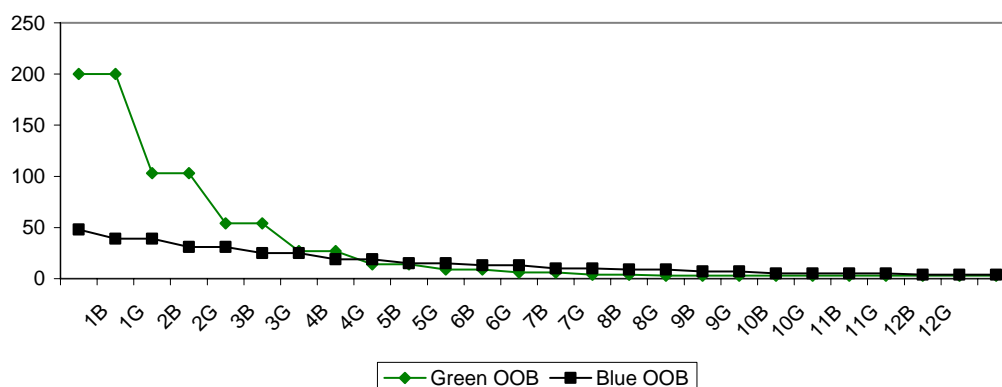


# ESCORTED BOMBERS ENGAGED BY SURFACE-TO-AIR MISSILE SITES

We developed two models of engagements between escorted bombers and SAM sites. The first, and more sophisticated of the two, considers engagements of four bombers, escorted by SEAD aircraft, with four SAM sites. The alternative model considers the situation in which the SEAD escorts engage the SAM sites before the sites engage the bombers, either because the SEAD escorts precede the bombers sufficiently or the SEADs have such low radar cross sections that the SAM sites' radars are unlikely to acquire them.

## Primary Model

The bombers are assumed to make a low-level attack, so that the SAMs have time to launch only two missiles before the bombers have dropped their bombs. An engagement with a SAM site commences when the site turns on its radar, attempting to lock on to a bomber (the site is assumed to ignore the escorts). When the site's radar comes on, the SEAD aircraft attempt to lock on to that emitter.

If the site locks first, it kills a bomber with probability $P_{ks}$. If the escort locks first, it kills the site with probability $P_{ke}$.

If the site misses, it turns off its radar and, if bombers are still available as targets, begins the engagement sequence again. The site also recommences the engagement sequence if the escort misses.

We assume that times to lock have the exponential distribution, for both the site and the escort. If the distributions' parameters are $\lambda_s$ for the site and $\lambda_e$ for the escort, straightforward calculations show that the probability that the site locks first is $\dfrac{\lambda_s}{\lambda_s + \lambda_e}$. It follows that the probabilities of the four possible outcomes of a single "round" of the engagement have the values shown in Table 2-1. The symbols given in that table for the outcomes will be convenient. We will also find it convenient to use the symbol $Q_{bk}$ for the quantity $1 - P_{bk}$, that is, the probability that no bomber was killed in a single round.

*Table 2-1. Outcome Probabilities*

| Outcome | Probability | Symbol |
|---|---|---|
| Site destroyed | $\dfrac{\lambda_e}{\lambda_s + \lambda_e} P_{ke}$ | $P_{sk}$ |
| Bomber destroyed | $\dfrac{\lambda_s}{\lambda_s + \lambda_e} P_{ks}$ | $P_{bk}$ |
| Site missed | $\dfrac{\lambda_s}{\lambda_s + \lambda_e}(1 - P_{ks})$ | $P_{sm}$ |
| Escort missed | $\dfrac{\lambda_e}{\lambda_s + \lambda_e}(1 - P_{ke})$ | $P_{em}$ |

With this information, we can develop the probabilities $p_0$, $p_1$, and $p_2$, which are, respectively, the probability of 0, 1, or 2 bombers killed in engagements with a single site, when at least two bombers are available as targets.

The probability of 0 kills is

$$p_0 = P_{sk} + (P_{sm} + P_{em})(1 - P_{bk}).\qquad\qquad 2\text{-}1$$

That is, the site makes no kill if it is destroyed in the first round, or, if it makes no kill in the first round but the engagement continues to the second round, and the site makes no kill in the second round.

The probability of 1 kill is

$$p_1 = (P_{sm} + P_{em}) P_{bk} + P_{bk} Q_{bk}.$$ 2-2

That is, the site either made 1 kill in the first round and no kill in the second, or, it made no kills in the first round, but there was a second round in which the site made a kill.

The probability of 2 kills is

$$p_2 = P_{bk}{}^2.$$ 2-3

We can now develop expressions for the probabilities $P_0$, $P_1$, $P_2$, $P_3$, and $P_4$, which are, respectively the probabilities of 0, 1, 2, 3, or 4 bombers killed. For the present, we make the restrictive assumption that the sites engage sequentially, so that the bombers "run a gauntlet" of separated sites. (We plan to treat the case of simultaneous engagements of bombers and sites in later work.)

We view the system after the results of each of the 4 sites separately. We will use the notation $P_n^i$ for the value of $P_n$ after the engagements of the $i^{th}$ site.

Obviously, after one site's engagement, $P_n^1 = pn$ for $n = 0$, 1, and 2; $P_3^1$ *and* $P_4^1$ are both zero, because the site has only two missiles.

After two sites' engagements, the probabilities of the numbers of kills are given straightforwardly by the convolution of the set ($p_0$, $p_1$, $p_2$) with itself.

This straightforward convolution process does not continue through three sites' engagements, however, because when three bombers are killed by the first two sites' engagements, only one bomber remains. The engagement of the third site cannot make two kills. Also, when the first two sites' engagements have made 4 kills, there is no engagement of the third or fourth site. The effect of these facts is to replace the simple convolution results with these values for $P_3^i$ *and* $P_4^i$ for $i = 3$ and $i = 4$:

$$P_3^i = Q_{kb} P_3^{i-1} + p_1 P_2^{i-1} + p2 P_1^{i-1}.$$ 2-4

$$P_3^i = Q_{kb} P_3^{i-1} + p_1 P_2^{i-1} + p2 P_1^{i-1}.$$ 2-5

Completing these calculations, we have values for $P_n = P_n^4$. For the parameter values of Table 2-2, Figure 2-3 shows the probability distribution of the number of bombers killed.

*Table 2-2. Parameter Values*

| $\lambda_e$ | $\lambda_s$ | $P_{ka}$ | $P_{ks}$ |
|:---:|:---:|:---:|:---:|
| 2 | 2/3 | 0.6 | 0.8 |

*Figure 2-3. Bomber Probability Distribution*
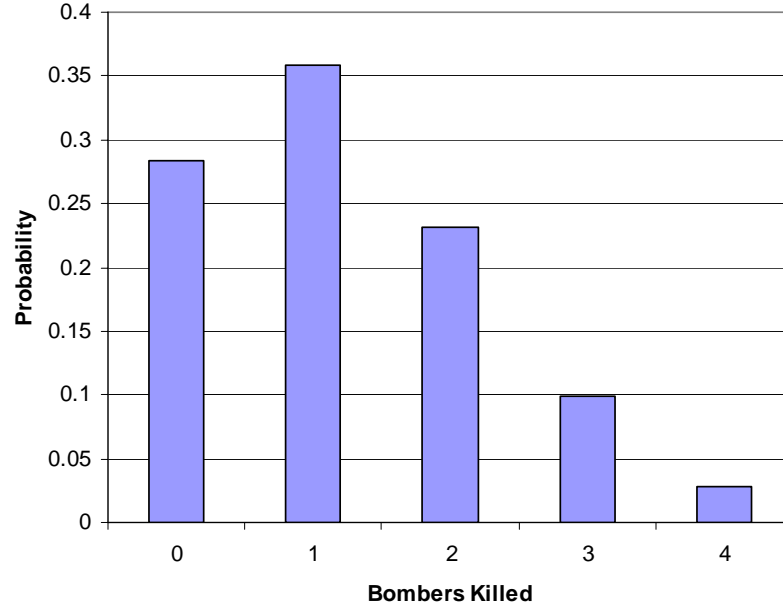


Although we made calculations with this model only for the case of exponential time-to-kill distributions, this restriction is not a property of the model itself. Other distributions may be used, and it is simple to change distributions. For example, to use the model when the time-to-kill distributions of both SAMs and escorts are those of the two-phase kill model discussed in an earlier report (*Stochastic Models of Air Superiority Engagements and Campaigns*), one need only replace the expression $\dfrac{\lambda_s}{\lambda_s + \lambda_e}$ in Table 2-1 by $P_{shipfirst}$:

$$P_{shipfirst} = 1 - \frac{\lambda_1^a \lambda_2^a}{\lambda_2^a - \lambda_1^a} \frac{1}{\lambda_2^s - \lambda_1^s} \left[ \frac{\lambda_2^s}{\lambda_1^a + \lambda_1^s} - \frac{\lambda_1^s}{\lambda_2^s + \lambda_1^a} - \frac{\lambda_2^s}{\lambda_2^a + \lambda_1^s} + \frac{\lambda_1^s}{\lambda_2^a + \lambda_2^s} \right] \qquad 2\text{-}6$$

and replace $\dfrac{\lambda_e}{\lambda_s + \lambda_e}$ by $1 - P_{shipfirst}$. In (2-6), the parameters $\lambda_1^a$ *and* $\lambda_2^a$ are, respectively, the parameters of the exponential distribution of the time for the escorts to lock onto a SAM site and of the exponential distribution of the time for the escorts to fire a missile, having locked on. Parameters $\lambda_1^s$ *and* $\lambda_2^s$ play the homologous role for the site.

# Alternative Model for Invulnerable SEAD Aircraft

It may happen that the SEAD escorts precede the bombers sufficiently that they engage the SAM sites before the sites can engage the bombers. It may also happen that the SEADs have such low radar cross sections that the SAM sites' radars are unlikely to acquire them. In this case, a simpler model is applicable. In this model, the only parameters are the number of missiles carried by the SEAD aircraft and the missiles' single-shot kill probability. If the aircraft have a total of $M$ missiles, and there are $N$ SAM sites, then straightforward analysis based on the usual "tree" diagram for binomial probabilities gives the probabilities of the engagement's outcomes.

To describe those outcomes, we may characterize the system state with the ordered pair *(m, n),* where $m$ is the number of missiles and $n$ the number of targets at the end of the engagement. Then, denoting the missiles *sspk* by $p$ and the quantity $1 - p$ by $q$, we have

$$P(M - N, 0) = B(N, N, p) \qquad \text{2-7}$$

$$P(M - N - j, 0) = B(N, N + j, p) - qB(N, N + j - 1, p), \quad 1 \le j \le M - N \qquad \text{2-8}$$

$$P(0, j) = B(N - j, M, p). \qquad \text{2-9}$$

Figure 2-4 shows an example of this model, when the SEAD aircraft fire 8 missiles with *sspk* = 0.6, against 4 SAM sites.

*Figure 2-4. SAM Probability Distribution*
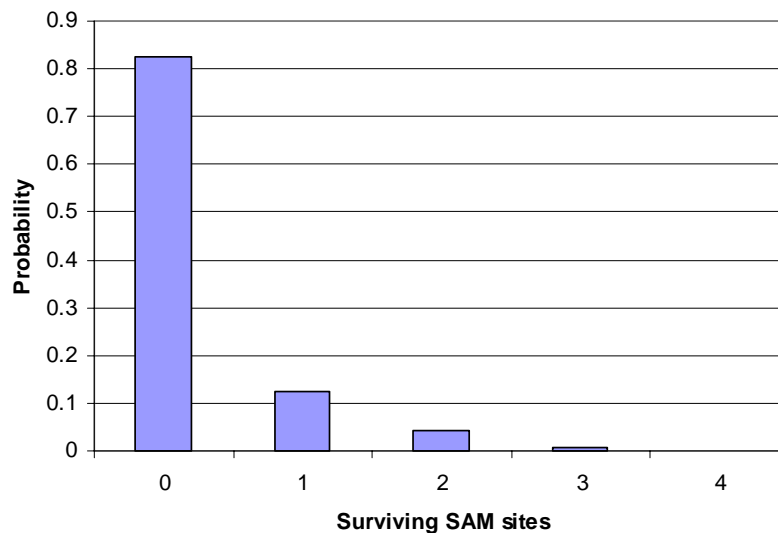
# MODEL OF ENGAGEMENTS BETWEEN A SEAD AIRCRAFT AND A MISSILE-FIRING SHIP

Some troop movements may be made by amphibious landing. In this case, missile-firing ships may escort troop-carrying ships. SEAD missions may then be required to permit tactical air attack of the troop carriers.

We model this situation by assuming that the attacking aircraft has four missiles (this number can be changed, of course) and that the ship has arbitrarily many missiles. The engagement begins when the aircraft illuminates the ship to fire a missile. At that time, the ship begins to develop a track on the aircraft.

We assume that the time for the aircraft to fire and the time for the ship to fire each have exponential distributions, with parameters $\lambda_a$ and $\lambda_s$, respectively. If the ship fires first, it kills the aircraft with probability $P_{ks}$; if the aircraft fires first, it kills the ship with probability $P_{ka}$.

If the ship misses, the opponents fight another round. If the aircraft misses and has not used all its missiles, the opponents fight another round. If the aircraft misses and has used all its missiles, it breaks off the engagement.

This model of the engagement leads to the discrete-time Markov process. We describe the system state with the ordered triple *(a, m, s)*, where $a = 1$ if the aircraft has not been destroyed, and $a = 0$ if it has been. The parameter *m* is the number of missiles on the aircraft. The parameter $s = 1$ if the ship has not been destroyed; otherwise $s = 0$.

The system begins in state $(1, 4, 1)$. At this time, four events are possible:

1. The ship locks first and kills the aircraft.

2. The ship locks first and shoots an unsuccessful missile.

3. The aircraft locks first and destroys the ship.

4. The aircraft locks firs, and shoots an unsuccessful missile.

We denote the probability of event *i* as $Q_i$. Table 2-3 shows the values of the $Q_i$ as functions of the engagement's parameters.

*Table 2-3. Event Probabilities*

| Event probability | Value |
|:---:|:---:|
| $Q_1$ | $\dfrac{\lambda_s}{\lambda_s + \lambda_a} P_{ks}$ |
| $Q_2$ | $\dfrac{\lambda_s}{\lambda_s + \lambda_a} (1 - P_{ks})$ |
| $Q_3$ | $\dfrac{\lambda_a}{\lambda_s + \lambda_a} P_{ka}$ |
| $Q_4$ | $\dfrac{\lambda_a}{\lambda_s + \lambda_a} (1 - P_{ka})$ |

In Table 2-4, we see that the system transitions from state (1, 4, 1) to state (0, 0, 1) with probability $Q_1$, to state (1, 4, 1) with probability $Q_2$, to state (1, 3, 0) with probability $Q_3$, and to state (1, 3, 1) with probability $Q_4$. Continued exploration of the system's states and transitions leads to the system's state transition matrix.

*Table 2-4. Aircraft versus Ship Transition Matrix*

| Initial state | Resultant state (aircraft,missile,ship) | | | | | | | | | |
|:---:|:---:|:---:|:---:|:---:|:---:|:---:|:---:|:---:|:---:|:---:|
| | 141 | 001 | 130 | 131 | 120 | 121 | 111 | 101 | 110 | 100 |
| 141 | $Q_2$ | $Q_1$ | $Q_3$ | $Q_4$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 001 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 130 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 131 | 0 | $Q_1$ | 0 | $Q_2$ | $Q_3$ | $Q_4$ | 0 | 0 | 0 | 0 |
| 120 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 121 | 0 | $Q_1$ | 0 | 0 | 0 | $Q_2$ | $Q_4$ | 0 | $Q_3$ | 0 |
| 111 | 0 | $Q_1$ | 0 | 0 | 0 | 0 | $Q_2$ | $Q_4$ | 0 | $Q_3$ |
| 101 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 110 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Note: We have omitted the parentheses and commas in the state symbols for the row and column labels to save space.

Let us call the state transition matrix *A*. We may then compute the engagement's outcome probabilities as

$$P_{(outcome)} = \lim_{n \to \infty} A^n e_1 \qquad \text{2-10}$$

where $e_1$ denotes the column vector with 1 in the top row and zeroes in every other row. That is, we know that the system begins in state (1, 4, 1), and the column of state probabilities after each transition is given by multiplying the column of probabilities before transition, by the matrix $A$.

We may, of course, see the values of P(outcome) by simply reading off the first row of $A^n$ for large $n$. Here, "large" means sufficiently large that the probabilities of all states except the absorbing boundary states—(0, 0, 1), (1, 3, 0), (1, 2, 0), (1, 0, 1), (1, 1, 0), and (1, 0, 0)—are small compared to 1.

The probability that the aircraft is destroyed is the limiting probability of state (0, 0, 1). The probability that the ship is destroyed is the sum of the limiting probabilities of states (1, 3, 0), (1, 2, 0), (1, 1, 0), and (1, 0, 0). The probability that the aircraft runs out of missiles and breaks off the engagement is the limiting probability of state (1, 0, 1).

The state transition matrix has a good deal of structure, so it is possible that some conclusions about the outcome probabilities can be reached analytically. We defer treating this possibility for later work, and content ourselves for now with numerical computations. The matrix multiplication routines of Microsoft Excel facilitate those computations.

Figure 2-5 shows the results of an example calculation for $\lambda_a = 0.2$, $\lambda_s = 0.125$, $P_{ka} = 0.8$, and $P_{ks} = 0.6$. With these parameters, both the ship's radar and the ship's missiles are distinctly inferior to those of the aircraft: the ship requires four times as long as does the aircraft to lock onto its target, and the *sspk* of the ship's missiles is only 75 percent of that of the aircraft's missiles. Nevertheless, as Figure 2-5 shows, the aircraft will loose about one-third of the engagements. (The probability of the aircraft's running out of missiles and breaking away, 0.00045, is too small to show in the figure.)

*Figure 2-5. Outcome Probabilities for SEAD/Ship Engagement*
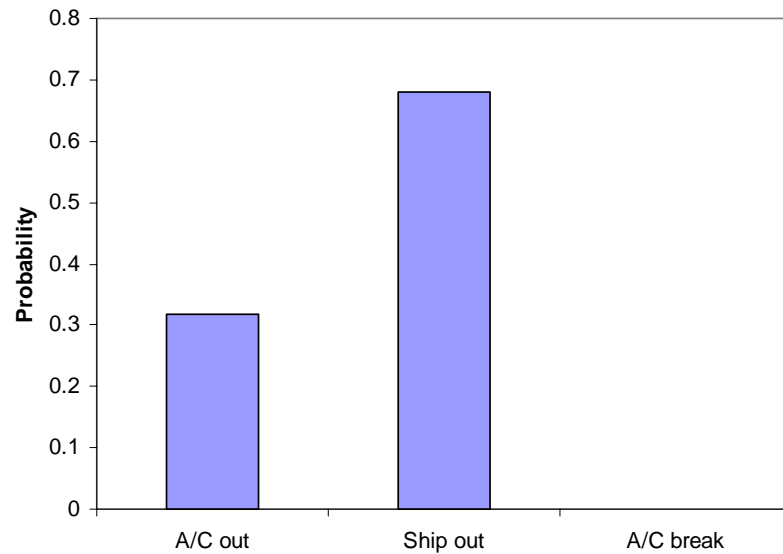


As one would expect, changing the parameters to give the ship better radar than the aircraft's radar turns the engagement strongly against the aircraft, even when the combatant's missiles have the same *sspk*. Figure 2-6 shows the outcome probabilities for $\lambda_a = 0.2$, $\lambda_s = 0.3$, $P_{ka} = 0.6$ and $P_{ks} = 0.6$.

*Figure 2-6. Outcome Probabilities, Ship's Radar Better Than Aircraft's*
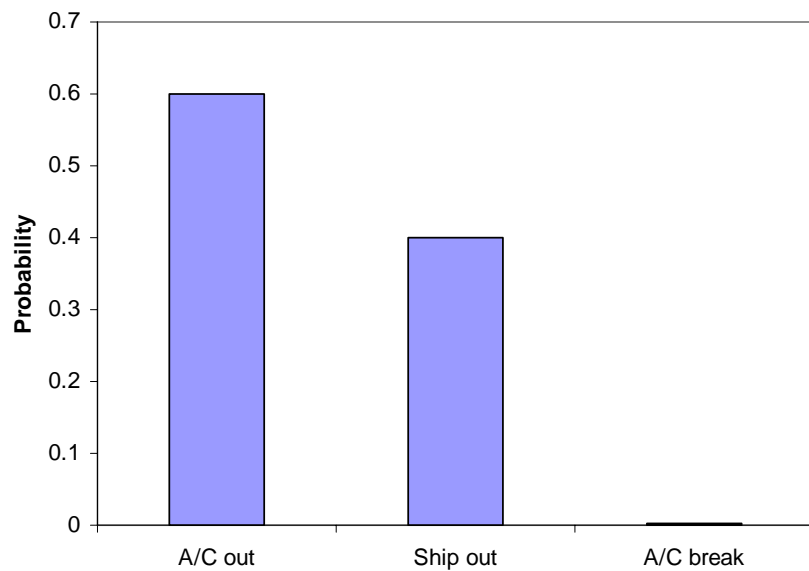


With *sspk*s as large as 0.6, the ship's advantage in having many more missiles than does the aircraft is not noticeable. The probability that the ship wins the engagement, which is the sum of the probabilities that the aircraft is killed and that the aircraft must break away, having run out of missiles, is not very different from

the probability that the aircraft is killed, because the probability of the aircraft's breaking away is so small.

For smaller *sspk*s, however, the ship's missile-carry advantage is much more noticeable. For example, if the two combatants' radars are equal, and their missiles both have an *sspk* of 0.2, then the probability that the aircraft breaks away is 20 percent, and the probability that the ship wins the engagement is 60 percent.

# TWO-PHASE KILL MODEL OF ENGAGEMENTS BETWEEN SEAD AIRCRAFT AND MISSILE-FIRING SHIPS

The model of the previous section treats a duel between one SEAD aircraft and one missile-firing ship. To model engagements between multiple SEAD aircraft and multiple ships, and to include the effects of an aircraft's maneuvering to break the lock of a ship's radar, we may use a two-phase kill model of the kind introduced in *Stochastic Models of Air Superiority Engagements and Campaigns*.[1] For completeness, we briefly describe this model here.

We consider an engagement in which $m$ aircraft engage $n$ missile-firing ships. We characterize the system state as $(m, i, n, j)$, were $m$ is the total number of SEAD aircraft, $i$ is the number of aircraft that have locked onto a ship, $n$ is the total number of missile-firing ships, and $j$ is the number of ships that have locked onto an aircraft.

In general, the seven events listed in Table 2-5 can change the system's state.

*Table 2-5. Ship Engagement Transition Events*

| Transition | Number |
|---|---|
| An aircraft can lock onto a ship | 1 |
| An aircraft can kill a ship that has locked onto an aircraft | 2 |
| An aircraft can kill a ship that has not locked onto an aircraft | 3 |
| An aircraft can break the lock of a tracking ship | 4 |
| A ship can lock onto an aircraft | 5 |
| A ship can kill a tracking aircraft | 6 |
| A ship can kill an aircraft that has not locked onto a ship | 7 |

We denote the probability that the system is in state $(m, i, n, j)$ at time $t$ by $P_{m,i,n,j}(t)$. If $M$ denotes the initial number of aircraft and $N$ the initial number of ships, then at the initial time, $P_{M,0,N,0}(0) = 1$ and all other $P_{m,i,n,j}(0)$ are zero. Let

---

[1] It seems unlikely that a ship can maneuver so well as to break the aircraft's lock, although this possibility can be introduced readily into an extended version of the model.

the rate at which the aircraft make lock be $k_{1a}$, the rate at which they make kills, having made lock, be $k_{2a}$, and the rate at which the aircraft break lock be $k_{xa}$. Let the rate at which the ships make lock be $k_{1s}$, and the rate at which they make kills, having made lock, be $k_{2s}$. The system's transient states evolve according to the equations:

$$
\begin{aligned}
\dot{P}_{m,i,n,j} = \quad & -\big((m-i)k_{1a} + ik_{2a} + jk_{xa} + (n-j)k_{1s} + jk_{2s}\big) & \text{(1)} \\
& + (m-i+1)k_{1a}P_{m,i-1,n,j} & \text{(2)} \\
& + (i+1)k_{2a}\frac{j+1}{n+1}P_{m,i+1,n+1,j+1} & \text{(3)} \\
& + (i+1)k_{2a}\frac{n+1-j}{n+1}P_{m,i+1,n+1,j} & \text{(4)} \\
& + (j+1)k_{ax}P_{m,i,n,j+1} & \text{(5)} \\
& + (n-j+1)k_{1s}P_{m,i,n,j-1} & \text{(6)} \\
& + (j+1)\frac{i+1}{m+1}k_{2s}P_{m+1,i+1,n,j+1} & \text{(7)} \\
& + (j+1)\frac{m-i+1}{m+1}k_{2s}P_{m+1,i,n,j+1}
\end{aligned}
$$

(2-11)

for $M_{min} < m \le M$, $0 \le i \le m$, $N_{min} < n \le N$, $0 \le j \le n$, where $M_{min}$ denotes the smallest number of aircraft with which the SEADs stay engaged. (The numbered terms in equation (2-11) correspond to the numbered transition events in Table 2-5.)

Rather than tracking the evolution of the system's absorbing boundary states, we find it convenient to track the evolution of the elements of the loss probabilities $Q(m, n)$, where $m$ is the number of aircraft lost and $n$ is the number of ships lost. Those evolution equations are

$$
\dot{Q}_{(M-m,N)} = \sum_{j=0}^{1}\sum_{i=0}^{m} ik_{a2}P_{m,i,1,j}
$$

(2-12)

and

$$
\dot{Q}_{(M-M_{min}+1,N-n)} = \sum_{j=0}^{n}\sum_{i=0}^{M_{min}} ik_{s2}P_{M_{min},i,n,j} \ .
$$

(2-13)

Figure 2-7 shows an example of the joint probability distribution Q(m, n) for the outcome of an engagement between four ships and four aircraft. The aircraft break away after sustaining two losses ($M_{min} = 3$). The ships cannot break off the engagement.

Figure 2-7. Joint Distribution of Outcome Probabilities



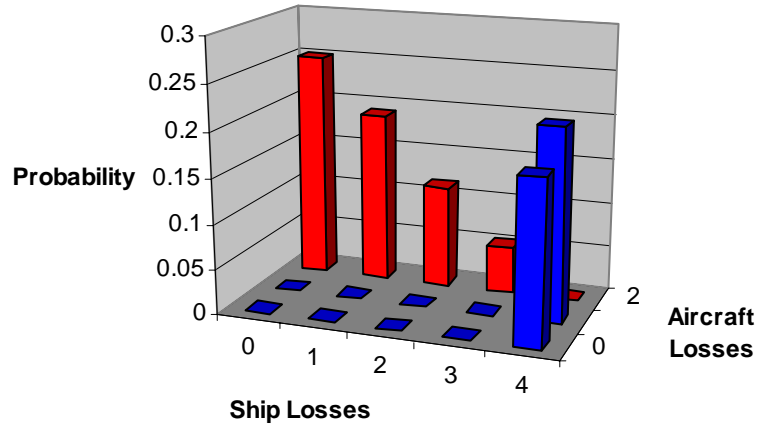*Figure 2-7. Joint Distribution of Outcome Probabilities*

Table 2-6 shows the values of the ship and aircraft parameters. The parameters describe an engagement in which the ships lock onto targets at twice the rate at which the aircraft lock onto targets; the aircraft and the ships both kill tracked targets at the same rate, and the aircraft break the ships' locks at twice the rate at which ships kill tracked targets.

*Table 2-6. Parameter Values*

| $k_{a1}$ | $k_{a2}$ | $k_{ax}$ | $k_{s1}$ | $k_{s2}$ |
|------|------|------|------|------|
| 0.5 | 2 | 4 | 1 | 2 |

# ENGAGEMENTS OF TACTICAL AIRCRAFT WHEN ONE COMBATANT'S AIRCRAFT CARRY ONLY TWO MISSILES

In *Stochastic Models of Air Superiority Engagements and Campaigns*, we reported on the development of  missile-tracking probabilistic models of engagements between flights of tactical aircraft. These models show the effects of varying missile loads on tactical aircraft engagements. Studies with these models indicated that, while finite missile loads did have some effect on engagement outcomes when aircraft carried six missiles, for many cases the results for models assuming infinitely many missiles did not differ too greatly from those of missile-tracking models.

Some of the aircraft of interest to TACAIR carry only two missiles, however. For these aircraft, it seems that missile-tracking models should be used. But when these aircraft's opponents carry six or more missiles, models tracking each combatant's missiles have 10-dimensional state vectors and involve very large numbers of states. This leads to cumbersome numerical work for engagement analyses.

We made a special engagement model for these cases. The model tracks missiles for the two-missile aircraft and assumes infinitely many missiles for their opponents. In this model, we called the two-missile aircraft the Red aircraft, and their opponents the Blue aircraft.

The state vector for this model is ($m, n_1, n_2$), where $m$ is the number of Blue aircraft, $n_1$ the number of Red aircraft with one missile, and $n_2$ is the number of Red aircraft with two missiles.

In general, the six events shown in Table 2-7 change the system's state.

*Table 2-7. Air-to-Air Transition Events*

| Transition event | Number |
|---|---|
| A Red carrying one missile makes a kill | 1 |
| A Red carrying two missiles makes a kill | 2 |
| A Red carrying one missile shoots a missile that does not make a kill | 3 |
| A Red carrying two missiles shoots a missile that does not make a kill | 4 |
| A Blue kills a Red that carries one missile | 5 |
| A Blue kills a Red that carries two missiles | 6 |

There is another event in which a Blue kills a Red with no missiles. In the present model, we avoid explicit accounting for this event with the assumption that a Red aircraft with no missiles is killed. This assumption could be changed, for example, with the assumption that a Red aircraft with no missiles breaks away.

The parameters of the model are $k_b$, the rate at which the Blue aircraft make kills; $k_r$, the rate at which Red aircraft make kills; and $p_k$, the single-shot kill probability of the Red missiles. A third parameter, $k_m$, the rate at which the Red aircraft fire missiles that miss, follows from $k_r$ and $p_k$ in this way: if the Red aircraft kill at the rate $k_r$, then they must fire at a rate $\mu$ such that

$$\mu p_k = k_r \Rightarrow \mu = \frac{k_r}{p_k} \Rightarrow k_m = \mu(1 - p_k) = k_r \frac{1 - p_k}{p_k}. \qquad \text{2-14}$$

Denoting the probability that the system is in state $(m, n_1, n_2)$ at time $t$ by $P_{m,n1,n2}(t)$, we find that the system's states evolve according to these equations:

$$\dot{P}_{m,n1,n2} = \quad -\left(mk_b + n\mu\right)P_{m,n1,n2}$$

$$1 \qquad + (n+1)k_r P_{m+1,n1+1,n2}$$

$$2 \qquad + (n2+1)k_r P_{m+1,n1,n2+1}$$

$$3 \qquad + (n1+1)k_m P_{m,n1+1,n2} H(m)$$

$$4 \qquad + (n2+1)k_m P_{m,n1,n2+1} H(m) \qquad\qquad 2\text{-}15$$

$$5 \qquad + mk_b \frac{n1}{n} P_{m,n1+1,n2}$$

$$6 \qquad + mk_b \frac{n2}{n} P_{m,n1,n2+1}$$

The number terms in equation (2-15) correspond to the transition events in Table 2-7. In (2-15), $n = n1 + n2$, and the Heaviside function $H(m)$ is zero if $m$ is less than zero, and one for all other values of $m$.

The system (2-15) of ordinary differential equations is solved subject to the initial condition

$$P_{M,0,N} = 1 \qquad\qquad 2\text{-}16$$

where M is the initial number of Blue aircraft and N is the initial number of Red aircraft, and with the conventions that $P_{m, n1, n2}$ is zero for all $(m, n1, n2)$ such that $m > M$ or the sum $n1 + n2$ is greater than $N$, and also is zero if any of $m, n1,$ or $n2$ is not positive.

We calculated outcome probabilities for eight Reds engaging four Blues, with $k_b = 3.7$, $k_r = 1.0$, and $p_k = 0.8$. Figure 2-8 and Figure 2-9 show that the loss distributions differ significantly from those of the classic stochastic Lanchester case. In particular, Red is much more likely to lose all eight aircraft when the limitation of two missiles is imposed.

*Figure 2-8. Blue Loss Distributions*
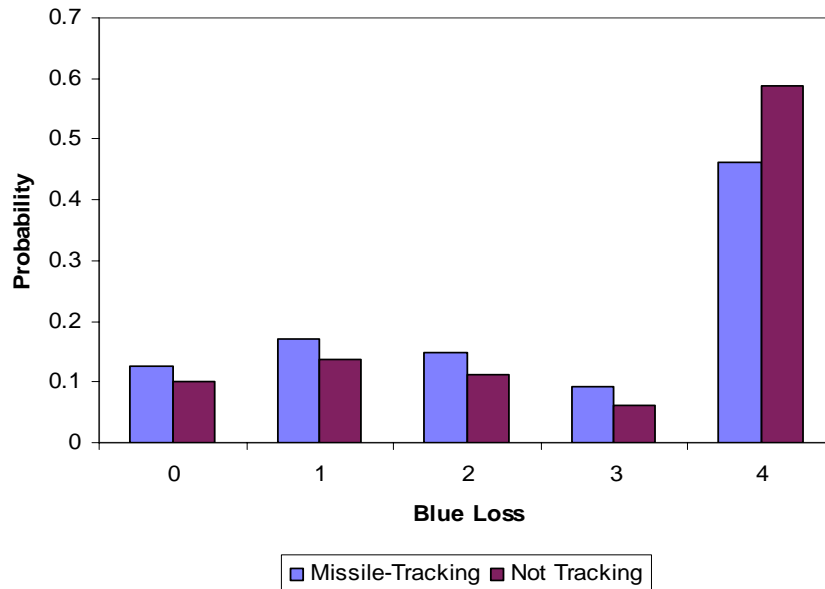


*Figure 2-9. Red Loss Distributions*
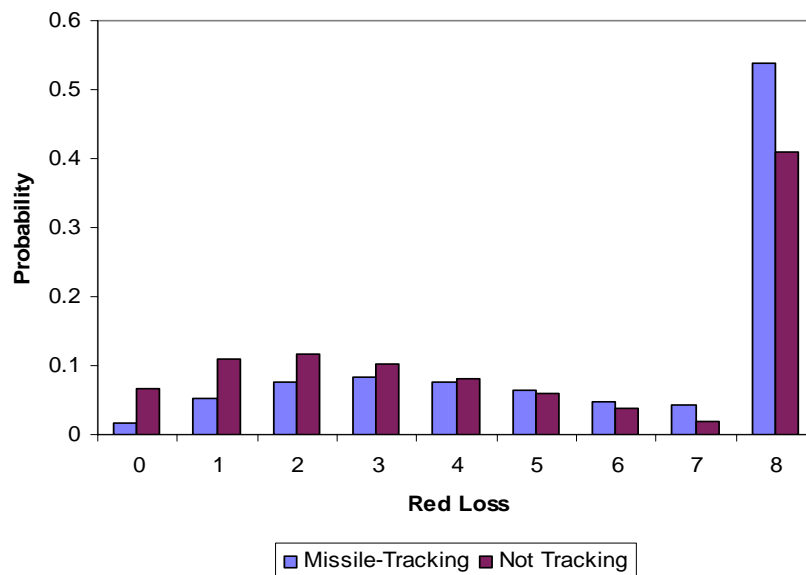


Note: The probability of the Blues winning the engagement is 54 percent.

# GENERALIZED ACTIVITY NETWORK MODELS OF COMBINED-FORCE ENGAGEMENTS

We began development of models treating engagements between combined forces. We expect such engagements to happen, for example, when defenders try to repulse an amphibious assault force of troop carriers escorted by missile-firing ships. Figure 2-10 shows such a combined force engagement.

*Figure 2-10. Defense against Amphibious Assault*

Legend:

| | |
|---|---|
| ▭ (red) | Missile-firing ship |
| ▭ (gray) | Troop carrier |

| | |
|---|---|
| → (red) | Covering tactical aircraft |
| → (blue) | Defenders engaging covering tactical aircraft |
| → (green) | SEAD engaging missile-firing ship |
| → (magenta) | Tactical aircraft engaging troop carriers |

In this combined-force engagement, an amphibious assault flotilla of four troop carrier ships escorted by two missile-firing ships and eight tactical aircraft is engaged by a defense package comprising four tactical aircraft whose mission is to eliminate the assault's tactical aircraft, two SEAD aircraft whose mission is to eliminate the missile-firing ships, and a set of tactical aircraft assigned to eliminate the troop carriers.

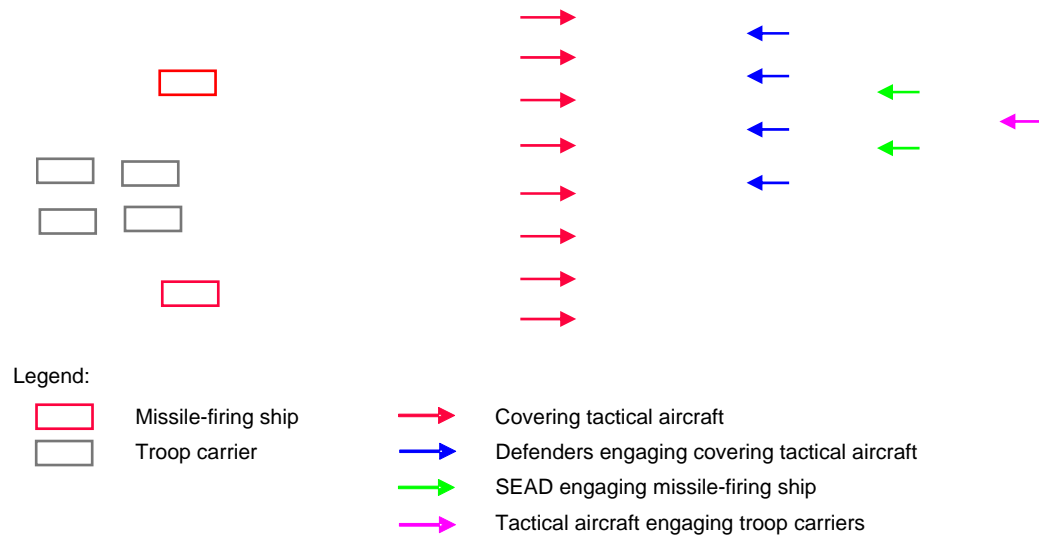As an illustrative example, Figure 2-11 gives a highly simplified generalized activity network (GAN) model of this combined force engagement.

*Figure 2-11. Generalized Activity Network Model*
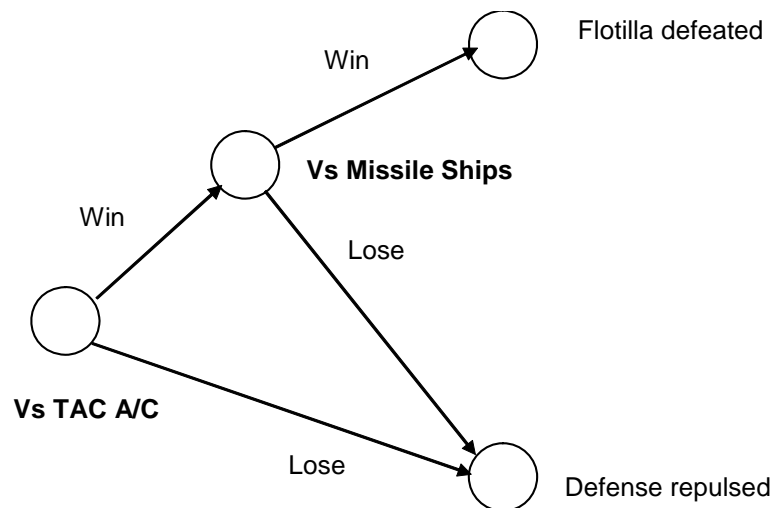


In this simple model, if the defenders assigned to defeat the flotilla's tactical air cover succeed, then the SEAD aircraft engage the missile-firing ships. Otherwise,

the defense is repulsed. If the SEAD aircraft defeat the missile-firing ships, the tactical aircraft targeted against the troop carriers succeed, and the flotilla is repulsed. If the SEADs do not succeed, the defense is defeated.

Models that we have developed generate the outcome probabilities for GANs such as this simple one, as well as for more complex ones. For example, if the engagement with the cover aircraft was the one of Figure 2-8 and Figure 2-9, and the two engagements of the SEAD aircraft and the missile-firing ships were the engagement of Figure 2-5, then the probability that the flotilla's air cover is destroyed is 54 percent, and the probability that both missile-firing ships are destroyed is 44 percent. This would lead to a 23 percent probability that the flotilla is repulsed.

# PROBABILITY ANALYSIS

We were asked to determine the probability, given eight different items and replacement after each draw, of drawing at least one of each of the eight items after 20 draws. We developed two solutions to the problem. The first was a derivation of the closed-form solution equation.

The closed form equation for the probability, given M different items and replacement after each draw, of drawing at least one of each of the M items after N draws is the following:

$$P_{M,N} = \frac{\sum_{i=0}^{i=M-1}(-1)^i C_{(M,i)}(M-i)^N}{M^N} \qquad \text{2-17}$$

where $C_{(M,i)}$ is the combination of the *M* items taken *i* at a time, i.e.,,

$$C_{(M,i)} = \frac{M!}{i!\,(M-i)!}. \qquad \text{2-18}$$

The second solution is a brute force spreadsheet model using generating functions for the combinatorial states and corresponding probabilities. Both methods were incorporated in an Excel workbook that was provided to TACAIR.

The answer to the specific question of drawing at least one each of eight items after 20 draws is 0.530558.

The calculations contained in the workbook can be easily expanded for arbitrary values of M and N. Figure 2-12 graphically shows the probabilities for M = 1 to 20 items and N = 1 to 20 draws. In the figure, M values are contained in the legend, and N values are on the abscissa.

*Figure 2-12. Probability, Given M Different Items with Replacement after Each Draw, of Drawing at Least One of Each of the M Items after N Draws*
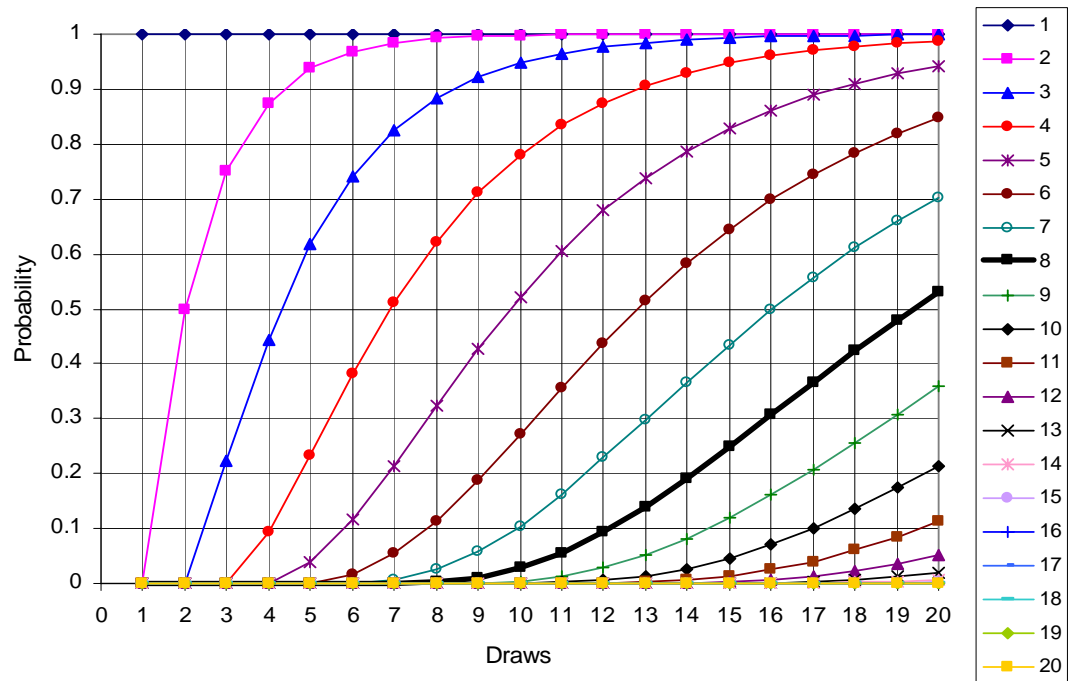


# KILL-RATES AND EXCHANGE RATIOS

In the course of our modeling efforts, confusion has arisen over the use of exchange ratios and kill-rates. This section describes our present way of inferring kill-rates, explains why we believe there's a better way to do this, and describes the better way.

## Background

Some workers describe the relative strengths of opposing forces in the context of a deterministic force-on-force Lanchester model, in the "weak combat" limit. Specifically, a Red force with $r_0$ members and kill-rate $k_r$ engages a Blue force with $b_0$ members and kill-rate $k_b$, in accordance with the model

$$\dot{r} = -k_b b$$
$$\dot{b} = -k_r r \qquad\qquad 2\text{-}19$$
$$r(0) = r_0; b(0) = b_0$$

and the engagement ends while the relative losses of both sides are small compared to one. In this case, the first two terms of the time-series solution of the initial value problem (2-19) dominate, and the exchange ratio $\rho$, that is, the ratio of Red's relative losses to Blue's, is given by

$$\rho = \frac{k_b b_0^2}{k_r r_0^2}.$$  2-20

Sometimes this result is stated in terms of the forces' strength ratio $\sigma$, in which the opponents' initial numbers appear linearly:

$$\sigma = \frac{\sqrt{k_b}\, b_0}{\sqrt{k_r}\, r_0} = \sqrt{\rho}.$$  2-21

Thus, given an exchange ratio $\rho$, and initial force values $r_0$ and $b_0$, one may infer the kill-rate ratio $k_b/k_r$ as

$$\frac{k_b}{k_r} = \rho \frac{r_0^2}{b_0^2}.$$  2-22

If one is given a strength ratio $\sigma$, then the kill-rate ratio follows as

$$\frac{k_b}{k_r} = \sigma^2 \frac{r_0^2}{b_0^2}.$$  2-23

Now, these relations are useful when data are given explicitly in terms of exchange ratios or strength ratios, and when it is known that the information is given in the context of the weak combat limit. In earlier work done for TACAIR, we assumed that strength ratios are given and that Red and Blue initial numbers are equal. That led to some large kill-rate ratios.

The relations are definitely not useful, however, if the given information is in terms of losses in deterministic engagements that go to completion, and that may involve breaking-away by either side. Furthermore, they are not useful if the information relates to statistics of probabilistic engagements.

# New Approach

We have been given engagement information that may be described reasonably accurately as the ratios of expected losses, in engagements of four Blues with eight Reds, when the Blues break away after sustaining 50 percent losses. To see the relation of that information to kill-rate ratios, we determined numerically the relation between kill-rate ratios and the ratio of Red expected loss to Blue expected loss, in four vs. eight engagements treated by the stochastic Lanchester

model, when Blues break away at 50 percent loss. Figure 2-13 displays the results.

*Figure 2-13. Ratio of Expected Loss vs. Kill-Rate Ratio in Four vs. Eight*
*Engagements when Blues Break Ways at 50 Percent Loss*
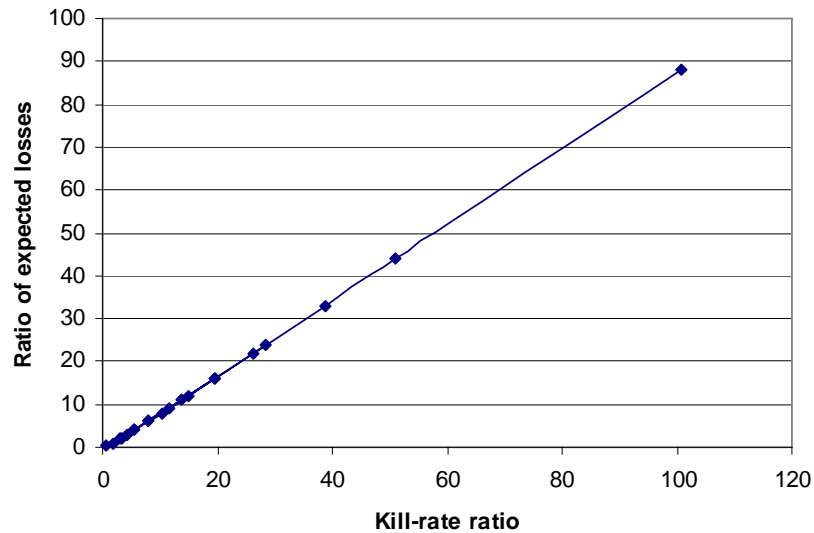


The results follow the linear function

$$Ratio\ of\ expected\ loses = 0.8802\ (Kill-rate\ ratio) - 0.9081 \qquad 2\text{-}24$$

with errors of less than 6 percent, for kill-rate ratios greater than about 3.75, corresponding to ratios of expected losses greater than about 2.25. The linear relation is not accurate for smaller kill-rate ratios.

When the linear relation is accurate, kill-rate ratios follow from ratios of expected losses as

$$Kill-rate\ ratio = 1.1362\ *(Ratio\ of\ expected\ losses) + 1.0318. \qquad 2\text{-}25$$

Values of kill-rate ratios corresponding to some of the observed ratios of expected losses are shown in Table 2-8.

*Table 2-8. Kill-Rate Ratios and Ratios of Expected Losses*

| &lt;Red loss&gt;/ &lt;Blue loss&gt; | Kill-rate ratio | &lt;Red loss&gt;/ &lt;Blue loss&gt; | Kill-rate ratio |
|---|---|---|---|
| 88 | 100.654 | 8 | 10.261 |
| 44 | 51.087 | 6 | 7.937 |
| 33 | 38.674 | 4 | 5.574 |
| 24 | 28.498 | 3 | 4.371 |
| 22 | 26.232 | 2.25 | 3.442 |
| 16 | 19.420 | 2 | 3.125 |
| 12 | 14.858 | 1 | 1.782 |
| 11 | 13.713 | 0.25 | 0.537 |

Inferring kill-rate ratios from ratios of expected losses directly from probabilistic engagement models leads to much smaller kill-rate ratios than those we have previously used. Believing this to be a more accurate model of the observations, we now use this new method until a better understanding of the engagements on which the data are based becomes available.

## Markov Model Comparison

We ran the NASA ASSIST and STEM Markov analysis tools to compare with the results above.[2] The conditions were four Blues vs. eight Reds with Blues leaving after two losses. We input the kill-rate ratios and used the results to calculate the exchange ratios, which is the opposite of the calculations above. The Markov model results agree exactly with the results above. Tables 2-9 through 2-11 show the input and output for each of three cases. Figures 2-14 through 2-16 show the corresponding details of the probabilities of Blue and Red losses. In each chart, the abscissa contains categories representing the numbers of Blue and Red aircraft lost, i.e., from left to right, B0–B2 Blue lost and then R0–R8 Red lost. The ordinate shows the loss probabilities of aircraft in each category.

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[2] ASSIST is the abbreviation for Abstract Semi-Markov Specification Interface to the SURE Tool, a high-order language tool to generate input files for both STEM and SURE Markov analysis tools. STEM is the abbreviation for Scaled Taylor Exponential Matrix, a Markov analysis tool. SURE is the abbreviation for Semi-Markov Range Estimator, a semi-Markov analysis tool not used here.

*Table 2-9. Blue and Red Loss Probabilities
for 0.537:1 Kill-Rate Ratio*

| Case 1 | |
| --- | --- |
| Input: B:R Kill-Rate Ratio | 0.537 |
| Output: B:R Exchange Ratio | 0.25 |

*Figure 2-14. Blue and Red Loss Probabilities for 0.537:1 Kill-Rate Ratio*
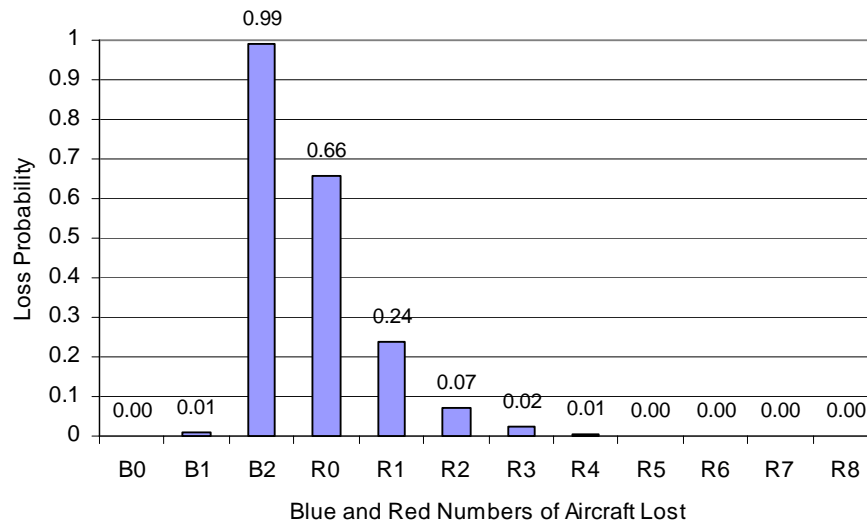


*Table 2-10. Blue and Red Loss
Probabilities for 10.261:1
Kill-Rate Ratio*

| Case 2 | |
| --- | --- |
| Input: B:R Kill-Rate Ratio | 10.261 |
| Output: B:R Exchange Ratio | 8.00 |

*Figure 2-15. Blue and Red Loss Probabilities for 10.261:1 Kill-Rate Ratio*



*Table 2-11. Blue and Red Loss Probabilities*
*for 100.654:1 Kill-Rate Ratio*

| Case 3 | |
|---|---|
| Input: B:R Kill-Rate Ratio | 100.654 |
| Output: B:R Exchange Ratio | 88.0 |

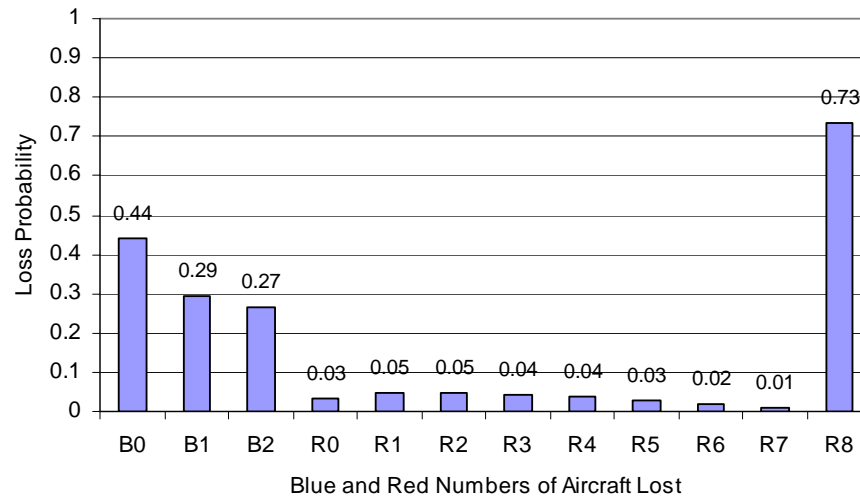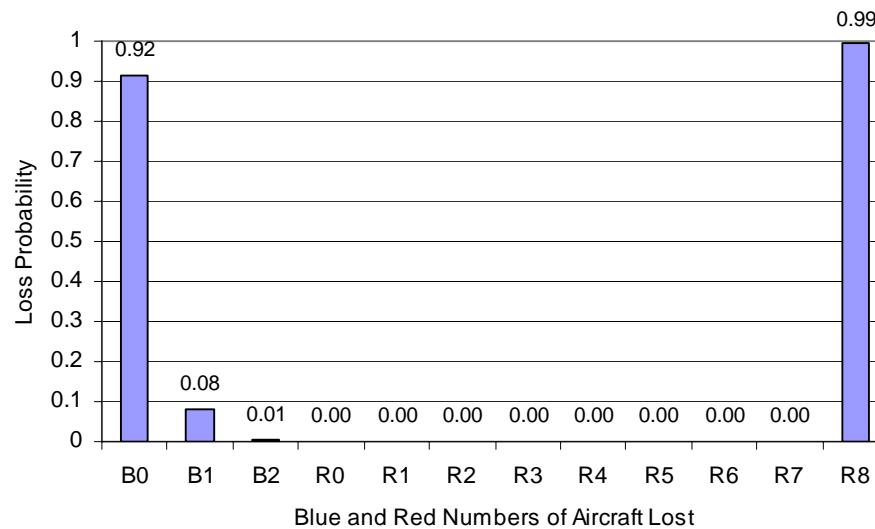*Figure 2-16. Blue and Red Loss Probabilities for 100.654:1 Kill-Rate Ratio*



2-25

# Chapter 3
# SLAACM as a Component in Integrated Analyses

Although individual SLAACM analyses certainly can be helpful, SLAACM may give even more useful results when operated as part of an integrated analysis that considers features not treated in SLAACM itself. Such analyses may involve the operation of other models, which may themselves be substantial packages.

It is always possible, of course, for an analyst to operate suites of models, passing data among the models of a suite by hand or by manually executed file transfers. This process is time-consuming and prone to error.

Applications packages now available provide convenient, automated operation of suites of models. LMI has worked with one of these, the Federated Intelligent Product Environment (FIPER).

FIPER is a result of a 4-year, $21.5 million project initiated to develop the next generation of product design and analysis technology. The National Institute of Standards and Technology Advanced Technology Program—along with Engine-ous Software (the prime software developer for FIPER components), General Electric, Goodrich, Parker-Hannifin, and OAI—provided funding for its development. Research partners also include Ohio University and Stanford University.

FIPER establishes an environment that streamlines the design of highly engi-neered products, integrating legacy and best-of-breed design and analysis tools through a web-enabled environment and supporting collaboration among geo-graphically distributed teams. The FIPER design addresses the problem of dis-tributed, collaborative product engineering. It integrates distributed models in a heterogeneous environment.

FIPER technologies are advanced state of the art and use standardized file formats and web-based communications protocols. Interface relationships between tools are discoverable, and more complex relationships are easily defined and invoked. FIPER uses standard XML to describe all data and component interfaces.[1] Tools developed to the FIPER component standard are plug-and-play compatible with other tools in the FIPER system. There are defined application package interfaces for building components to provide compatibility with individual modeling and

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[1] A component is a collection of objects that provide a cohesive set of services to a client. Each of the objects that make up the component can be developed independently. Also, each set of components can be developed independently. Objects are software bundles of data and procedures. The procedures acting on the data are known as methods. The state of an object is determined by its data, while the behavior of an object is defined by its methods.

simulation applications. Finally, FIPER is designed to be web based, allowing integration of models that operate on heterogeneous platforms over a network or the web.

As an experiment to test FIPER's usefulness, and using LMI's own resources, the team made an exploratory example of using SLAACM as part of an integrated cost/performance optimization study, in which FIPER was the integrating agent. Figure 3-1 describes the problem treated in the study.

*Figure 3-1. Optimization Study*

- Requirement:  75 Blue fighter a/c defeat optimal assault from mix of 792 a/c, allowing no more than 1,100 tons of bombs dropped by "leakers," with 80% confidence that there are no more than 45 Blue losses.
- Current fighter:  1,200 tons dropped, 56 losses
- Alternative I:  Improve airframe and propulsion ($12B)
    - 1,067 tons dropped, 42 losses
- Alternative II:  Improve radar (Cost and outcome are functions of improvement)
- Alternative III:  Improve missile sspk (Cost and outcome are functions of improvement)

The team used available LMI models of radar cost and missile cost. The models were roughly related to actual cost data. *The cost models are intended only to give an example of the use of SLAACM in an integrated environment, and the conclusions of the study are not necessarily valid for any actual radars, missiles, or aircraft.*

As shown in Figure 3-2, we made the test using SLAACM to assess the warfighting implications—specifically, the 85 percent-confidence value of Blue losses, and the expected value of tons of bombs dropped—of changes in missile single-shot kill probability (sspk) and radar power-aperture product (PA). We used FIPER to drive five models: a radar model, a radar cost model, a missile cost model, a kill-rate model reflecting the effect on kill-rate ratios of changes in sspk and PA, and SLAACM.

*Figure 3-2. Integrated Analysis Configuration*
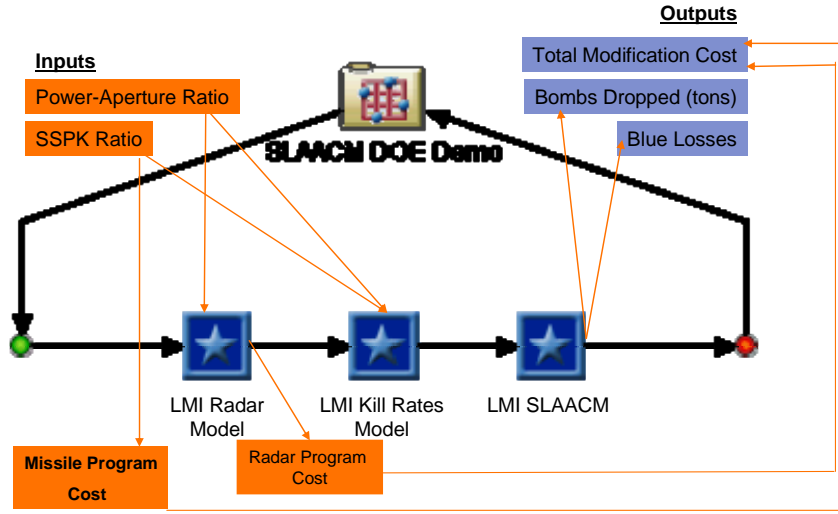


Here is the way we inferred changes in kill-rate ratios, resulting from changes in sspk and power-aperture product. We considered a two-phase kill process. The first phase is target acquisition, and we associated improvement in the radar's power-aperture product with that phase. Specifically, we modeled the changed first phase rate $k_{1\,new}$ as proportional to the original first phase rate $k_{1\,old}$ and to the ratio of the new power-aperture product, $PA_{new}$, to the old power-aperture product, $PA_{old}$:

$$k_{1\,new} = k_{1reference} \frac{PA_{new}}{PA_{old}}. \qquad\qquad 3\text{-}1$$

In a two-phase kill process the rate of the second phase, $k_2$, is equal to the firing rate $\mu$ times sspk. Accordingly, we modeled the effect of changed sspk by

$$k_{2\,new} = k_{2old} \frac{sspk_{new}}{sspk_{old}}. \qquad\qquad 3\text{-}2$$

Now, in the two-phase kill model, the time to make a kill is the sum of two exponentially distributed random variables, one with parameter $k_1$ and the other with parameter $k_2$. The mean of a sum is the sum of the means of its terms, so the mean time to make a kill, $T_{mean}$, is given by

$$T_{mean} = \frac{1}{k_1} + \frac{1}{k_2}. \qquad\qquad 3\text{-}3$$

The mean rate $k_{mean}$ of the time to make a kill is the reciprocal of $T_{mean}$. Accordingly, we found the ratio of new kill rate $k_{new}$ to old kill rate $k_{old}$ as

$$\frac{k_{new}}{k_{old}} = \frac{k_{1\,new}k_{2\,new}}{k_{1\,old}k_{2\,old}}\frac{k_{1\,old} + k_{2\,old}}{k_{1\,new} + k_{2\,new}}.$$

3-4

We adjusted the kill-rate ratios of all the Red aircraft to the new, modified Blue aircraft by the ratio calculated with (3-4). *We made this rough-and ready adjustment only for an example study to test using SLAACM in an integrated model. We would revisit this model an any actual study.*

In the test, we used FIPER's design-of-experiment (DOE) mode. In this mode, the FIPER user specifies a range of sspk ratios and PA ratios, and FIPER determines, with a DOE procedure, a set of N (sspk, PA) pairs to map out the system's behavior.

We instructed FIPER to treat a set of 64 (sspk, PA) pairs, and completed the example study by identifying the least-cost member of the set of feasible (sspk, PA) pairs, that is, the ones that met the requirements of no more than 1,100 tons of bombs dropped, and 85 percent confidence that Blue would have no more than 45 losses. The results are shown in Figure 3-3.

*Figure 3-3. Example Study Results: Feasible Points and Solution*
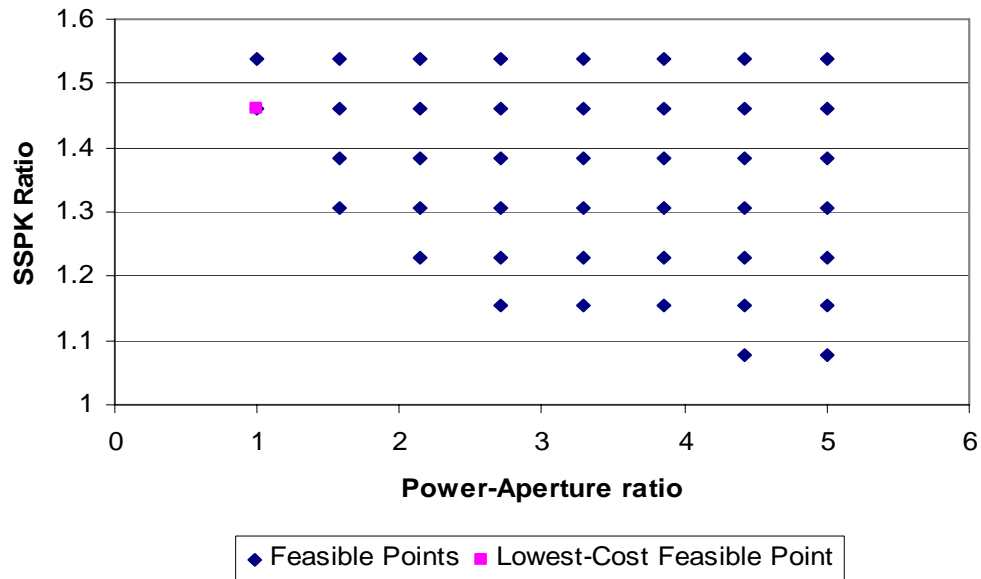


It is interesting that the optimal result, for the cost models of the study, would be to modify the missile, and not the airplane or the radar. That conclusion depends, of course, on the very rough cost models and kill rate model used, and is not necessarily correct for any actual aircraft, radar, or missile.

We concluded from this experiment that SLAACM may be used helpfully in linked suites of models, for more general studies than those SLAACM alone can address.

# Chapter 4
# Runway Interdiction Analysis

This chapter documents a mathematical analysis of runway interdiction by sub-munition (bomblet) weapons. It has three sections. In the first, we describe the problem and methods of solution. In the second, we develop a solution for the case in which the runway width is not more than twice the minimum required operating width. In the third section, we extend the solution to three times the minimum width.

## THE BASIC PROBLEM

Figure 4-1 illustrates the runway interdiction problem. Attacking a runway of width 2W, a missile delivers N submunitions with impact points distributed uniformly over the circle of radius R, centered at the missile's impact point $X_I$. Each submunition destroys a circle of diameter d, centered on its impact point. The runway is still usable if it has an undamaged lane of width r, running parallel to its centerline. Otherwise the runway is closed. Missile impacts within distance R of the runway's ends are unlikely to close it, and they are ignored in this problem statement. Also for this reason, only abscissas of the missile and submunition impact points are relevant to the interdiction problem. We refer all abscissas to the runway centerline.

*Figure 4-1. Runway Interdiction*
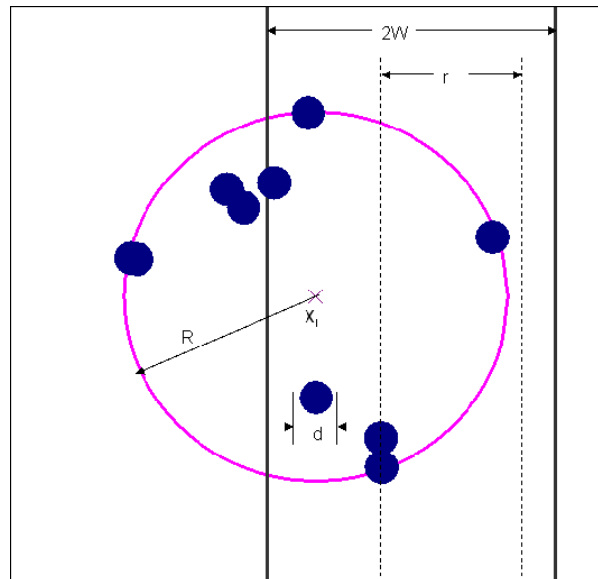
## Description of Lanes

Label a lane on the runway with the abscissa of its left-hand end. The lane e is the set [e, e + r].  The runway is closed if some part of every lane with e in [−W, W − r] is damaged by the submunitions; otherwise it is still usable.

## Lanes Damaged by a Submunition

A submunition impacting with abscissa $x_i$ damages lanes with e in the set $E(x_i)$, given by

$$E(x_i) \equiv \left[\max(x_i - d/2 - r, -W), \min(x_i + d/2, W - r)\right]. \qquad 4\text{-}1$$

## A Formal Statement of the Interdiction Problem

A formal statement of the interdiction problem can be given with the $E(x_i)$. It is simply

$$\bigcup_1^N E(x_i) \supset [-W, W - r]. \qquad 4\text{-}2$$

The $x_i$ are the abscissas of N draws from the bivariate uniform distribution of the submunitions' impact points. Let that distribution be $g(x; X_I)$ (an expression for $g(x; X_I)$ is in a following section). The joint probability distribution of the $\{x_i\}_1^N$ is $g(x_1, X_I)\, g(x_2, X_I)\ldots g(x_N, X_I)$, and, in principle, the probability that equation (4-2) holds can be expressed as an integral of that probability density function (pdf) over the appropriate region. Determining that region seems difficult, however, at least for N greater than 2 or 3. The use of order statistics may help develop explicit calculations expressing the probability that (4-2) is met.

## Order Statistics

The order statistics of the $\{x_i\}_1^N$ are the numbers $Y_1, Y_2, \ldots, Y_N$ obtained by ordering the $x_i$. That is, $Y_1$ is the smallest of the $x_i$, $Y_2$ is the next smallest, and so on up to $Y_N$, which is the largest of the $x_i$. Order statistics are of considerable interest in several contexts, and they have been extensively studied.[1]

In terms of the $Y_i$, the requirement that the rightmost lane is closed is that there must be some $Y_k$ such that

$$Y_k + d/2 \geq W - r. \qquad 4\text{-}3$$

---

[1] See, for example, H. L. Harter, *The Chronological Annotated Bibliography of Order Statistics* (American Sciences Press, 1991).

That is, some submunitions must close the lane with $e = W - r$. Let $Y_R$, the left-most impact that closes the rightmost lane, be the smallest of these.

Similarly, some submunitions must close the leftmost lane, that is, there must be some $Y_j$ such that

$$Y_j - d/2 - r \le -W .\qquad\qquad 4\text{-}4$$

Let $Y_L$, the rightmost impact that closes the leftmost lane, be the largest of these. Then, to close all the lanes, the set of lanes closed by $Y_L$, $Y_{L+1}$, …, $Y_R$ must have no gaps. The requirement that the lanes closed by $Y_{i-1}$ overlap those closed by $Y_i$ is

$$Y_{i-1} + d/2 > Y_i - d/2 - r \qquad\qquad 4\text{-}5$$

or

$$Y_i - Y_{i-1} < d + r .\qquad\qquad 4\text{-}6$$

Thus a statement of the interdiction problem in terms of the order statistics of the $\{x_i\}_1^N$ is that there be a sequence $Y_L$, $Y_{L+1}$, …, $Y_R$ of order statistics such that

$$
\begin{array}{ll}
1. & Y_L - d/2 - r \le -W \Rightarrow Y_L \le -(W - r - d/2) \\
2. & Y_R + d/2\mu W - r \Rightarrow Y_R \ge W - r - d/2 \\
3. & Y_i - Y_{i-1} \le d + r, \quad L+1 \le i \le R
\end{array}
\qquad 4\text{-}7
$$

The joint probability distribution function of the $Y_i$ is known to be

$$f(Y_1, Y_2, \ldots,, \ldots, Y_N) = g(Y_1; X_I)\, g(Y_2; X_I) \ldots g(Y_N; X_I), \ \forall\ Y_1 \le Y_2 \le \ldots \le Y_N \qquad 4\text{-}8$$

and this fact will be helpful in calculating the probability that the interdiction conditions are met. Using the order statistics $Y_i$ instead of the $x_i$ makes it possible to consider sets of variables with consecutive indices.

## Solution of the Interdiction Problem with Order Statistics

It appears possible to solve the interdiction problem by treating a mutually exclusive and exhaustive set of sequences $Y_L$, $Y_{L+1}$, …, $Y_R$ that meet the requirements (4-7). Here is a description of that process.

The smallest possible value for $Y_L$ is $Y_1$. Depending on the values of W and r, there is a smallest number of damage regions that can close the runway. Let that number be $N_c$. Then one member of our set of mutually exclusive and exhaustive

sequences is $Y_1$, $Y_2$, …, $Y_{Nc}$. Conditions 7.1, 7.2, and 7.3 applied to that sequence explicitly define a region of $Y_1$, $Y_2$, …, $Y_N$ space, and, in principle, one can compute the integral of the distribution function given in (4-8) over that region. In a similar way, one can compute the probabilities of all other sequences meeting conditions (4-7), that is, those for which $Y_R$ is $Y_{Nc+1}$, $Y_{Nc+2}$, … $Y_N$. The process continues by taking $Y_L = Y_2$, $Y_3$, … and so on, through the largest possible value for $Y_L$, which is $Y_{N+1-Nc}$.

The sequences generated in this way are mutually exclusive, and they exhaust the set of sequences satisfying conditions (4-7). Adding their probabilities gives the probability of interdiction. This solution is, perhaps, computationally feasible when $N_c$ is not too large, i.e., two or three for damage diameters of 10 feet, impacting a runway that is 150 feet wide.

# The Bivariate Circular Uniform Distribution and Its Marginal Distributions

These considerations will give us an explicit expression for $g(x; X_I)$. Measured from the missile's impact point $X_I$, the coordinates $(x, y)$ of a submunitions' impact point are uniformly distributed on a circle of radius R. Accordingly, the bivariate distribution of $(x, y)$ is

$$f(x,y) = \begin{cases} \dfrac{1}{\pi R^2}, & x^2 + y^2 < R^2 \\ 0, & \text{else} \end{cases}. \qquad \text{4-9}$$

The marginal distribution of the impact point's abscissa, $g(x)$, is given by

$$g(x) \equiv \int_{-R}^{R} f(x,y)dy = \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2}, \; -R \le x \le R. \qquad \text{4-10}$$

The cumulative marginal distribution of the impact point's abscissa, $G(x)$, is given by

$$G(x) \equiv \int_{-R}^{x} g(x)dx = \frac{2}{\pi R^2}\int_{-R}^{x}\sqrt{R^2-x^2}\,dx = \frac{1}{2} + \frac{1}{\pi}\left(\sin^{-1}(x/R) + \frac{x\sqrt{R^2-x^2}}{R^2}\right). \text{4-11}$$

By the symmetry of $f(x, y)$, the marginal distribution of ordinates is given by $g(y)$.

Let $x_I$ denote the abscissa of $X_I$. Then the function $g(x; X_I)$ is given by

$$g(x;X_I) = g(x-x_I) = \begin{cases} \dfrac{2}{\pi R^2}\sqrt{R^2-(x-x_I)^2}, & -R \le x-x_I \le R \\ 0, & \text{else} \end{cases}. \qquad \text{4-12}$$

# GENERATING SAMPLES OF BIVARIATE UNIFORM RANDOM VARIABLES

This task is made more convenient by expressing f(x, y) with polar coordinates $(r, \theta)$. To do this, we want a function $\hat{f}(r, \theta)$ such that

$$\iint\limits_{(x,y) \in H} f(x, y) \, dx \, dy = \iint\limits_{(r,\theta) \in H} \hat{f}(r, \theta) \, dr \, d\theta \qquad 4\text{-}13$$

where H is any region of the plane. Now,

$$\iint\limits_{(x,y) \in H} f(x, y) \, dx \, dy = \iint\limits_{(r,\theta) \in H} f(X(r, \theta), Y(r, \theta)) \, r \, dr \, d\theta \qquad 4\text{-}14$$

where

$$X(r, \theta) \equiv r \cos(\theta); \quad Y(r, \theta) \equiv r \sin(\theta). \qquad 4\text{-}15$$

Thus

$$\hat{f}(r, \theta) = \begin{cases} \dfrac{r}{\pi R^2}, & 0 \le r \le R \\ 0, & \text{else} \end{cases} \qquad 4\text{-}16$$

that is, $\hat{f}(r, \theta)$ is independent of $\theta$, and, as a function of r, $\hat{f}(r, \theta)$ is proportional to r.

To get pairs $(r, \theta)$ corresponding to draws from a bivariate uniform distribution, one may choose values of $\theta$ from a uniform distribution on $[0, 2\pi]$. To obtain the accompanying value of r, one may invert the cumulative distribution

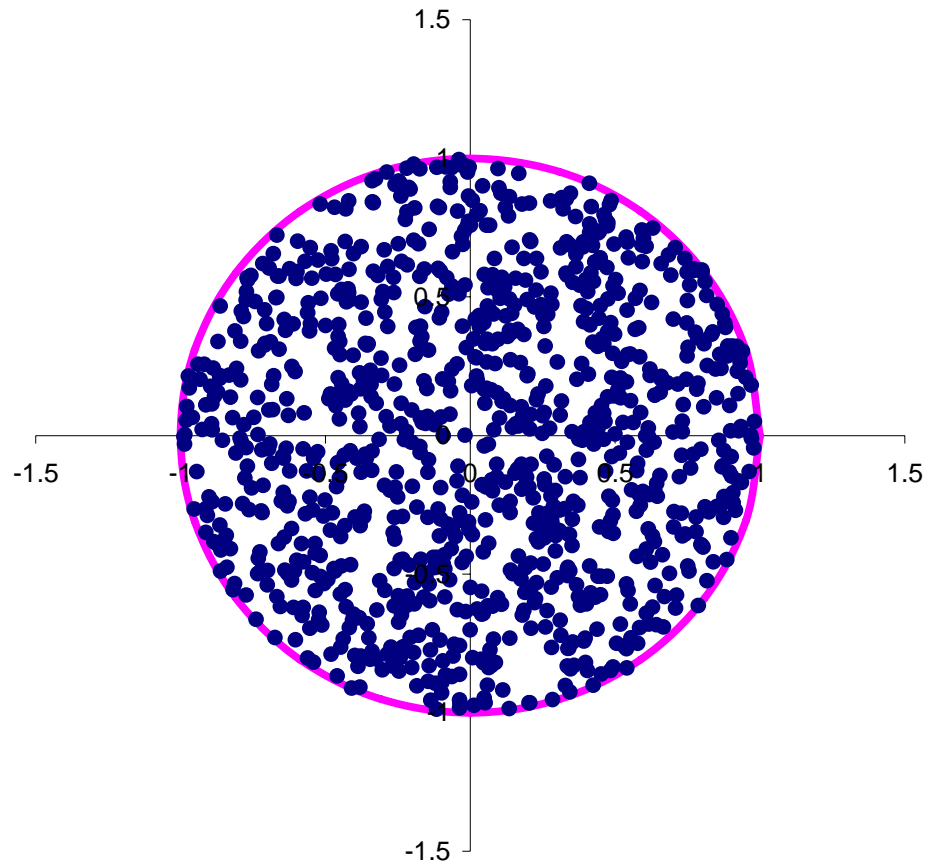$$2\pi \int_0^r \frac{u}{\pi R^2} \, du = \frac{r^2}{R^2} \qquad 4\text{-}17$$

and set

$$r = R\sqrt{x} \qquad 4\text{-}18$$

where x is uniformly distributed on [0, 1]. Figure 4-2 shows a sample of 1,000 points drawn from a bivariate uniform distribution.

*Figure 4-2. Sample from Bivariate Circular Uniform Distribution*

# SOLUTION FOR 2X CASE

The analysis below is based on the following assumptions and observations:

◆ The description of the basic airport runway interdiction problem.

◆ The realization that the problem can be addressed by working on the marginal distribution across the runway. "Marginal distribution" in this context means the distribution along one dimension, x, reduced from the distribution in two dimensions, x and y.

We try here to offer some idea on how to calculate the "passable" probability for the entire runway, which is the ultimate probability for the runway interdiction problem. We first consider passable probability at a given point.

For the derivation of the passable probability at a given point, we follow the approach of working on the marginal distribution of the bomblets perpendicular to the runway centerline and assuming the radius of the bomblets r=0.
Let

◆ x be the abscissa established perpendicular to the runway centerline, and x = 0 correspond to the left edge of the runway (x = 0 here corresponds to x = −W in the section on the basic problem)

◆ f(x) be the marginal probability density function of bomblets along x

◆ n be the total number of bomblets contained in a bomb or a missile (this corresponds to the N parameter in the basic problem)

◆ r be the impact radius of each bomblet

◆ G be the minimum gap between the bomblets or between a bomblet and the edges of the runway that allows flight operations

◆ $N(x_1,x_2)$ be the number of bomblets in the interval $(x_1,x_2)$

◆ $P_k(x_1,x_2) = Pr\{ N(x_1,x_2) = k \}$, or the probability of k bomblets in the interval $(x_1,x_2)$ ($P_0$ means there is no bomblet in the interval, not the probability at the origin, $x_0$).

Then, for a small Δx, the probability of having one bomblet in an interval from x to x+Δx is

$$P_1(x, x + \Delta x) \cong n * f(x) * \Delta x .$$   4-19

Also, for a sufficient small Δx, the probability of having more than one bomblet in an interval from x to x+Δx is

$$\sum_{j=2}^{n} P_j(x, x + \Delta x) = o(\Delta x)$$   4-20

where o(Δx) means a lower order than Δx.

Let $x_0$ be a fixed point. The probability of having no bomblets in the interval from the origin, $x_0$, to x+Δx is

$P_0(x_0,x+\Delta x) = Pr\{ N(x_0,x+\Delta x) = 0 \}$

$= Pr\{ N(x_0,x) = 0, N(x, x+\Delta x) = 0 \}$

$= Pr\{ N(x_0,x) = 0 \} * Pr\{ N(x, x+\Delta x) = 0 \mid N(x_0,x) = 0 \}$

$= P_0(x_0,x) * Pr\{ N(x, x+\Delta x) = 0 \mid N(x_0,x) = 0 \}$

$= P_0(x_0,x) * (1 - Pr\{ N(x, x+\Delta x) = 1 \mid N(x_0,x) = 0 \} )$

$$= P_0(x_0, x) \left[ 1 - \frac{n * f(x)\Delta x}{1 - \int_{x_0}^{x} f(t)dt} \right].$$  4-21

(The $t$ is an arbitrary variable and the integral including $f(t)$ is a cumulative distribution function over $x_0$ to x.)

Equation (4-21) holds because of (4-19) and the conditioning of the probability. Integrating (4-22) from $x_0$ to $x_0+G$, will yield the passable probability at point $x_0$. In other words, there is no bomblet in the interval $(x_0, x_0+G)$; the probability of 0 hits for bombs containing n bomblets is

$$s_n(x_0) = P_0(x_0, x_0 + G) = \exp \left[ -n \int_{x_0}^{x_0+G} \frac{f(x)}{1 - \int_{x_0}^{x} f(t)dt} dx \right].$$  4-22

Now we extend the analysis to the problem of the whole runway. The marginal distribution f(x) can take any functional form in the following derivation, including the half-circle marginal density function from the original assumption of the 2-dimensional uniform distribution within a circle.

For ease of derivation, we first assume r = 0, and then adjust the intervals. For r = 0, we say the runway is passable at $x_0$ if there is a minimum gap of G from $x_0$, or there is no bomblet in $[x_0, x_0+G]$; we call its complement "blocked." We say the runway is passable in $[x_0, x]$ if it is passable at any point of the interval, and its complement is blocked. Let Q(x) be the probability that the runway is passable in the interval of $[x_0, x]$, then

Q(x+Δx) = Pr{ passable in $[x_0, x+Δx]$ }

= Pr{passable in $[x_0,x]$ ∪ passable in $[x, x+Δx]$ }

= Pr{ passable in $[x_0,x]$ } + Pr{ passable in $[x, x+Δx]$ ∩ blocked in $[x_0,x]$ }

= Q($x_0$, x)+ Pr{ passable in $[x, x+Δx]$ ∩ blocked in $[x_0,x]$ }

= Q($x_0$, x)+ (1 - Q($x_0$, x) ) * Pr{ passable in $[x, x+Δx]$ | blocked in $[x_0,x]$ },

because

Pr{ passable in $[x, x+Δx]$ ∩ blocked in $[x_0,x]$ }

= Pr{ blocked in $[x_0,x]$ } * Pr{ passable in $[x, x+Δx]$ | blocked in $[x_0,x]$ }

$$= (1 - Q(x_0, x)) * Pr\{ \text{ passable in } [x, x+\Delta x] \mid \text{blocked in } [x_0,x] \}.$$

Let $b_m$, $m = 1, 2, \ldots, n$, be the events that $[x_0, x]$ is "blocked" and there are exactly m bomblets in $[x_0, x]$ (Note, for some values of m, the event of $b_m$ can be null, depending on the parameters, but such cases do not change the following derivation.)

Continuing

$$Pr\{ \text{ passable in } [x, x+\Delta x] \mid \text{blocked in } [x_0,x] \}$$

$$= Pr\{ \text{ passable in } [x, x+\Delta x] \mid b_1 \cup b_2 \ldots \cup b_n \}$$

$$= ( Pr\{ \text{ passable in } [x, x+\Delta x] \cap b_1 \} + \ldots,$$
$$+ Pr\{ \text{ passable in } [x, x+\Delta x] \cap b_n \} )/Pr\{ b_1 \cup b_2 \ldots \cup b_n \},$$

where the last equation holds because the events $b_m$, $m = 1, 2,., n$, are mutually exclusive.

For all $m \in \{1, 2, \ldots, n\}$, and by the definition of $b_m$, every point of in $[x_0, x]$ is blocked, including the end point x. If point x is blocked, therefore, there must be at least one bomblet in the interval of $[x, x+G]$. However, for the event that the runway is passable in $[x, x+\Delta x]$, it means there is no bomblet in the interval of $[x+\Delta x, x+\Delta x+G]$. Thus, for a small $\Delta x$, the intersection of the events that the runway is blocked from $x_0$ up to x, and passable at $x+\Delta x$, leads to the conclusion that there is one bomblet in $[x, x+\Delta x]$, and there is no bomblet in $[x+\Delta x, x+\Delta x+G]$. Let $p_m(x)= Pr\{b_m\}$, to first order in $\Delta x$, we have

$$Pr\{ \text{ passable in } [x, x+\Delta x] \cap b_m \}$$

$$= p_m (x) * Pr\{ \text{ passable in } [x, x+\Delta x] \mid b_m \}$$

$$= p_m (x) * m * f (x) * \Delta x * s_{n-m}(x + \Delta x )$$

where the last equation holds since the two events are independent except the density of seeing one bomblet in $[x, x+\Delta x]$ is now given by $m*f(x) \Delta x$, and $s_{n-m}(x + \Delta x)$ is the probability that the point $x + \Delta x$ is passable. This is a similar idea to the one we used to derive the point passable probability, except that the number of bomblets in the interval $[x_0, x]$ makes difference.

Let $p_0(x) = Pr\{ b_1 \cup b_2 \ldots \cup b_n \}$, which is actually $1 - Q(x)$. We then have the equation

$$Q(x + \Delta x) = Q(x) + (1 - Q(x)) * \sum_{m=1}^{n} \frac{m p_{m-1}(x) s_{n-m}(x + \Delta x)}{p_0(x)} * f(x)\Delta x \qquad \text{4-23}$$

or

$$\frac{dQ(x)}{dx} = (1-Q(x)) * \sum_{m=1}^{n} \frac{mp_{m-1}(x)s_{n-m}(x)}{p_0(x)} * f(x).$$ 

4-24

Now, we have a differential equation for $Q(x)$, where the subscript m−1 follows from the fact there is one bomblet in $(x, x+\Delta x)$. With the initial condition $Q(x_0) = s_n(x_0)$, which is the probability that an aircraft can "pass" at point $x_0$, we have an initial value problem for $Q(x)$.

From the equation above, one can see that $Q(x)$ is a non-decreasing function and is not greater than 1, which are the necessary conditions. Once this fact is established, we can then rewrite the equation above, taking the fact that $p_0(x) = 1 - Q(x)$, as follows

$$\frac{dQ(x)}{dx} = \sum_{m=1}^{n} mp_{m-1}(x)s_{n-m}(x) * f(x).$$

4-25

$s_{n-m}(x)$ is the probability that the runway is passable at point x when there are m bomblets in $[x_0, x]$. By its definition, there is no bomblet in $[x, x+G]$. In other words, none of the remaining $n - m$ bomblets falls in $[x_0, x+G]$. For a similar derivation as we did for the point passable probability calculation, we have

$$s_{n-m}(x) = \exp\{-(n-m)\int_{x}^{x+G} \frac{f(u)}{1-\int_{x}^{u} f(t)dt} du\}$$

4-26

If we know the probabilities $p_m(x)$, for m=1, 2, …,n, then we should be able to solve the differential equation for $Q(x)$. We have not figured out a way to solve the problem in general. Since the event $b_m$ is not only the case that there are m bomblets in the interval $[x_0, x]$, but also the event that the bomblets are scattered in such a way that there is no gap bigger than G to let an aircraft pass, $p_0 + \Sigma p_m \neq 1$ in general.

If, however, $|x - x_0| <= G$, then any number of m ($>=1$) bomblets falling in the interval $[x_0, x]$ will automatically make the interval blocked regardless of their relative locations inside. Under this condition, $p_m(x)$ is the probability that there are m bomblets in $[x_0, x]$, which is given by

$$p_m(x) = C_m^n \left(\int_{x_0}^{x} f(t)dt\right)^m \left(1-\int_{x_0}^{x} f(t)dt\right)^{n-m}.$$

4-27

Thus, at least theoretically, we should be able to compute the passable probability for a section of a runway $[x_0, x_0+G]$ for any $x_0$. In other words, if the runway width is no more than 2G, then we should be able to compute the passable probability for the entire runway. If we have to consider the size of the bomblet, this condition can be verified to be 2G+2r. If the runway width is bigger than that, we do not have a complete solution. We can, perhaps, use curve-fitting, or some kind

of bounding based on different values of $x_0$ for a segment of a runway of 2G+2r width. We may also use the fact that the passable probability is the highest when $x_0$ is close to the tails of f(x) and that it is a non-decreasing function of additional runway.

# EXTENDING TO THE 3X SOLUTION

We now extend the analysis to runways of width 3 times the gap (G).

We have previously established the following:

$$\frac{dQ(x)}{dx} = \sum_{m=1}^{n} m p_{m-1}(x) s_{n-m}(x) * f(x) \qquad 4\text{-}28$$

where Q(x) is the probability that the runway is passable in [$x_0$, x], and $s_{n-m}(x)$ is the probability that the runway is passable at point x when there are m bomblet(s) in [$x_0$, x], which is given by

$$s_{n-m}(x) = \exp\left\{ -(n-m)\int_x^{x+G} \frac{f(u)}{1 - \int_x^u f(t)dt} du \right\}, \qquad 4\text{-}29$$

and $p_m(x)$, for m=1, 2, …, n, is the probability that there are m bomblets in the interval [$x_0$, x] and that interval is also blocked.

As stated earlier, the difficulty of applying equation (4-28) is to figure out the blocked probability in [$x_0$, x]. With this in mind, and for the easier expression, we will work with the following variant of (4-28), to expand our results from 2 times to 3 times of the needed takeoff width $G$, which lies in the calculation of $p_m(x)$, for $G < x - x_0 < 2G$:

$$\frac{dQ(x)}{dx} = \sum_{m=0}^{n-1} (m+1) p_m(x) s_{n-m-1}(x) * f(x). \qquad 4\text{-}30$$

In general, the event $b_m$—there are m bomblets in [x0,x] and [x0,x] is blocked—is the equivalent of the following two events: there are m bomblets in [$x_0$,x], and those m bomblets will make [$x_0$,x] blocked. In general, we have

$$p_m(x) = \alpha^* \beta \qquad 4\text{-}31$$

where

$\alpha$ is the probability that m bomblets are in the interval [$x_0$,x], and

$\beta$ is the conditional probability that the m bomblets will make $[x_0,x]$ blocked.

From our earlier analysis, we have

$$\alpha = C_m^n \left( \int_{x_0}^x f(t)dt \right)^m \left( 1 - \int_{x_0}^x f(t)dt \right)^{n-m} , \qquad \text{4-32}$$

which holds regardless of the value of x.

The difficulty of applying (4-31) lies in the expression of $\beta$. (Fortunately, for x − $x_0$<G, $\beta \equiv 1$, because the assumed bomblet at $[x, x+\Delta x]$ will always block any point in $[x_0,x]$.) For G<x- $x_0$<2G, we have the following.

◆ If m=0, $\beta$=0, since the assumed bomblet at $[x, x+\Delta x]$ can block only the interval $[x-G, x]$, which makes $[x_0,x]$ passable because any point in $[x_0,x-G]$ is passable.

◆ If m=1, $[x_0,x]$ is blocked if and only if the bomblet is in $[x-G, x_0 +G]$, which leads to

$$\beta = \frac{\int_{x-G}^G f(t)dt}{\int_{x_0}^x f(t)dt} . \qquad \text{4-33}$$

In order to compute $\beta$ when m>1, we need to divide the interval $[x_0,x]$ into three disjoint intervals: A=$[x_0,x-G]$, B=$[x-G, x_0+G]$, and C=$[x_0+G, x]$. In general,

$$\beta = \chi \cdot \delta + \varepsilon * \phi \qquad \text{4-34}$$

where

$\chi$ is the conditional probability that there is at least one bomblet in B, given m bomblets in $[x_0,x]$,

$\delta$ is the conditional probability that in $[x_0,x]$ is blocked given the event that there is at least one bomblet in B,

$\varepsilon$ is the conditional probability that there is no bomblet in B, given m bomblet(s) in $[x_0,x]$, and

$\phi$ is the conditional probability that $[x_0,x]$ is blocked given the event that there is no bomblet in B.

The probability of no bomblet in B, or of all m bomblets either in A or C, is given directly by

$$\varepsilon = \left( \frac{\int_A f(t)dt + \int_C f(t)dt}{\int_{A+B+C} f(t)dt} \right)^m = \left( \frac{\int_{x_0}^{x-G} f(t)dt + \int_G^x f(t)dt}{\int_{x_0}^x f(t)dt} \right)^m ,$$  4-35

and the probability of at least one bomblet in B is given by

$$\chi = 1 - \varepsilon = 1 - \left( \frac{\int_{x_0}^{x-G} f(t)dt + \int_G^x f(t)dt}{\int_{x_0}^x f(t)dt} \right)^m .$$  4-36

For the same logic as m=1, we have $\delta = 1$.

The only term that we need to work now in Eq. 30 is to figure out $\phi$. Let $i, j$ be the number of bomblets in A and C, respectively, and $i + j = m$. In general,

$$\phi = \sum_{i=0}^m \gamma_i \bullet \eta_i$$  4-37

where

$\gamma_i$ is the conditional probability that there are i bomblets in A, given the event that all m bomblets are either in A or C, and

$\eta_i$ is the conditional probability that $[x_0, x]$ is blocked, given that there are i bomblets in A and m-i bomblets in C.

For i = 0, 1, …, m, $\gamma_i$ is given by the probability of the binomial distribution as

$$\gamma_i = C_i^m \left( \frac{\int_A f(t)dt}{\int_{A+C} f(t)dt} \right)^i \left( \frac{\int_C f(t)dt}{\int_{A+C} f(t)dt} \right)^{m-i}$$  4-38

or

$$\gamma_i = \frac{C_i^m}{\left( \int_{A+C} f(t)dt \right)^m} \left( \int_A f(t)dt \right)^i \left( \int_C f(t)dt \right)^{m-i} .$$  4-39

The following logic attempts to figure out $\eta_I$ in (4-33). First, if $i = 0$, then $\eta_I$ implies A is empty with no bomblet, which, when combined with the fact that all m

bomblets are either in A or C, means all m bomblets are in C only, or both A and B and empty. Since A and B are contiguous, it means there is no bomblet in $[x_0, x-G] + [x-G, x_0+G] = [x_0, x_0+G]$. For point $x_0$, its gap to the next bomblet (in C) is then more than G, which means it is open. Thus, $\eta_0 = 0$.

Similarly, $\eta_m = 0$. Thus, (4-33) can be rewritten as

$$\phi = \sum_{i=1}^{m-1} \gamma_i \bullet \eta_i \ . \qquad\qquad \text{4-40}$$

When $i>0$, $j>0$, the largest possible gaps for the bomblets within A or C, respectively, are the lengths of A or C, which are less than G. It implies that the $[x_0, x]$ is blocked if and only if the distance of the smallest in C and the largest in A is no more than G.

For any bomblet in A, let $g_A$ and $G_A$ be its conditional density function and CDF, respectively. For any bomblet in C, let $g_C$ and $G_C$ be its conditional density function and CDF, respectively. Since they are the conditional probability functions, we have

$$g_A(y) = \frac{f(y)}{\int_A f(t)dt} \qquad\qquad \text{4-41}$$

if $y \in A = [x_0, x-G]$; $g_A(y) = 0$, if $y \notin A$;

$$g_C(z) = \frac{f(z)}{\int_C f(t)dt} \qquad\qquad \text{4-42}$$

if $z \in C = [x_0+G, x]$; $g_C(z) = 0$, if $z \notin C$.

And thus,

$$G_A(y) = \frac{\int_{x_0}^{y} f(t)dt}{\int_A f(t)dt} \qquad\qquad \text{4-43}$$

if $y \in A = [x_0, x-G]$, and

$$G_C(z) = \frac{\int_{x-G}^{z} f(t)dt}{\int_C f(t)dt} \qquad\qquad \text{4-44}$$

if $z \in C = [x_0+G, x]$.

Let

$H_A$ be the CDF of the largest bomblet in A,

$h_A$ be the density function of the largest bomblet in A,

$H_C$ be the CDF of the smallest bomblet in C, and

$h_C$ be the density function of the smallest bomblet in C.

Because the bomblets are independent, and by their definitions, we have

$$H_A(y) = 1 - [1 - G_A(y)]^i = 1 - \left[ 1 - \frac{\int_{x_o}^y f(t)dt}{\int_A f(t)dt} \right]^i \qquad \text{4-45}$$

$$H_C(z) = [G_C(z)]^j = \left[ \frac{\int_G^z f(t)dt}{\int_C f(t)dt} \right]^j \qquad \text{4-46}$$

and

$$h_A(y) = i[1 - G_A(y)]^{i-1} g_A(y) \qquad \text{4-47}$$

$$h_C(z) = j[G_C(z)]^{j-1} g_C(z) . \qquad \text{4-48}$$

Since the bomblets are independent, so are the smallest in C and the largest in A. Thus,

$$\eta_i = \int_{z-y \le G} h_A(y)h_C(z)dydz = \int_{x_0}^{x-G} h_A(y)dy \int_{x_0+G}^{y+G} h_C(z)dz = \int_{x_0}^{x-G} h_A(y)H_C(y+G)dy . \qquad \text{4-49}$$

After integrating by steps, (4-45) can alternatively be expressed as

$$\eta_i = 1 - \int_{x_0}^{x-G} h_C(y+G)H_A(y)dy . \qquad \text{4-50}$$

With (4-49) or (4-50) for $\eta_i$, one can figure out $\phi$ in (4-34), which leads to the computation of (4-30), which further makes it possible to compute (4-31). Thus, theoretically, one is able to compute the block probability $p_m(x)$ up to 2 times of G. In other words, we should be able to compute the passable probability $Q(x)$ of a runway whose width does not exceed 3G. Most of the terms involve one-dimensional integration.

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The report describes analytical tasks performed for OSD/PA&E TACAIR during the period from September 2004 through September 2005. The report describes use of and extension to the stochastic Lanchester Air-to-Air Campaign model (SLAACM) probabilistic model of campaigns for air superiority between two opponents, described in previous reports. The current report describes air combat mathematical analyses of air-to-air defense, surface-to-air missile defense against escorted bombers, air attacks against a missile-firing ships, air-to-air combat with limited missile inventories, and generalized models of combined force engagements. In a separate section, the report discusses use of SLAACM as a component in integrated resource analysis. The final section of the report describes derivations of mathematical models for analyzing runway interdiction by cluster bombs.

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